Differential forms on free and almost free divisors

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To Egbert Brieskorn

Abstract

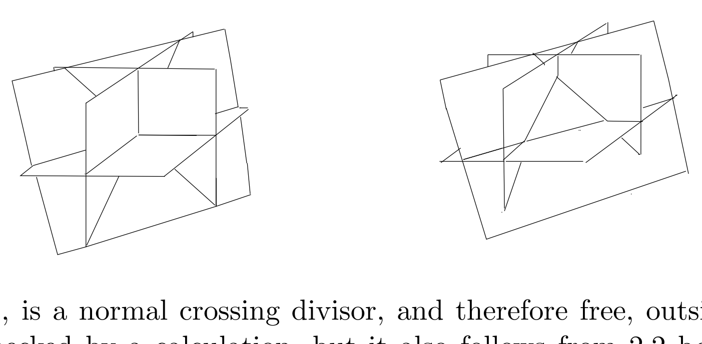
We introduce a variant of the usual Kähler forms on singular free divisors, and show that they enjoy the same depth properties as Kähler forms on isolated hypersurface singularities. Using these forms it is possible to describe analytically the vanishing cohomology in families of free divisors, in precise analogy with the classical description for the Milnor fibration of an ICIS, due to Brieskorn and Greuel. This applies in particular to the family $\{D(f_\lambda)\}_{\lambda\in\Lambda}$ of discriminants of a versal deformation $\{f_\lambda\}_{\lambda\in\Lambda}$ of a singularity of a mapping.

1 Introduction

The hypersurface D in the complex manifold X is a *free divisor* if the $\mathcal{O}_X$-module $\mathrm{Der}(\log D)$ of germs of vector fields on X which are tangent to D, is locally free(see [31]). Examples of free divisors include smooth hypersurfaces, normal crossing divisors, reflection arrangements ([30]) and discriminants of right-left stable maps with source dimension not less than target dimension ([22], 6.13).

In many contexts, one is give a divisor D_0 which is free outside 0, and a family $D \to \mathbb{C}$ with special fibre D_0, in which the general fibre is free. Typically, D_t grows homology classes which vanish when t returns to 0.

A familiar example of this phenomenon is shown in the following pair of diagrams. In the first, we see four planes passing through 0 in 3-space, of which each three are in general position.



Their union, D_0, is a normal crossing divisor, and therefore free, outside 0. It is not free at 0; this can be checked by a calculation, but it also follows from 2.2 below. In the second

picture, one of the planes has been shifted off 0, and the union, D_t, now has a non-trivial second homology class, carried by the tetrahedron one can see in the centre of the picture (in fact the inclusion of the real D_0 and D_t in their complexification, induces a homotopy-equivalence, so these are "good real pictures"). Since D_t is now a normal crossing divisor (everywhere), it is free. One can think of D_t as a kind of singular Milnor fibre of D_0 — we have not smoothed D_0 (as in "classical" singularity theory); instead we have freed it.

Another large class of examples: if $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ (with $n \geq p$) is a germ of analytic map stable outside 0 (equivalently, having "finite $\mathcal{A}$-codimension") then the discriminant $D(f_0)$ is free outside 0; if f_t is a stable perturbation of f_0 then $D(f_t)$ is a free divisor, which carries homology classes which vanish when t returns to 0.

In [9] Damon introduced a definition of *almost free divisor*, and a suitable category of deformations, which encompass these examples and many others. Using Morse-theoretic techniques introduced in [12], he is able to calculate the rank of the vanishing homology. In this paper we describe the vanishing cohomology in the singular Milnor fibres of almost free divisors in terms of differential forms.

For a reasonable description, the usual Kähler forms Ω^p_D (the definition is recalled at the start of Section 3) are not suitable: there are too many of them. It is well known that if x is a singular point of a divisor D then $\Omega^k_{D,x}$ has torsion for dim $D + 1 \geq k \geq$ dim $D -$ dimD_{sing}. The torsion gives important geometrical information: indeed, if the n-dimensional divisor D has an *isolated* singularity at x then the torsion submodule of $\Omega^n_{D,x}$ has length equal to the Tjurina number, and thus less than or equal to the Milnor number, the rank of the vanishing homology (and equal to it if D has a weighted homogeneous defining equation). In this case one can think of torsion n-forms as forms which are idle (evaluate to 0 on every n-vector) on (D, x), but which come to life on the Milnor fibre of (D, x). The torsion submodule of $\Omega^n_{D,x}$ thus encodes information on the vanishing geometry of the Milnor fibre. On a singular free divisor or almost free divisor D the singular subspace has codimension 1, and thus if dim $D > 1$, all the modules of Kähler forms (except for Ω^0) will have torsion submodules of infinite length. In the deformations described by Damon, almost free divisors become free, but are not smoothed, and thus infinite dimensional spaces of Kähler k-forms will wait in vain for their vanishing k-vectors.

The obvious thing to do to remedy this is to kill the torsion forms on a free divisor D (as free divisors are the stable objects in Damon's theory). Doing so, we obtain modules which we denote $\check{\Omega}^k_D$. The definition of the modules $\check{\Omega}^k_{D_0}$ of forms on an almost free divisor D_0 then follows by a standard universal procedure — embed D_0 in a suitable free divisor D, as the fibre of an ambient submersion to a smooth space S, and then take the quotient of the module of relative forms $\check{\Omega}^k_{D/S}$ defined in the standard way from $\check{\Omega}^k_D$. In fact Damon defines an almost free divisor $D_0 \subset V$ as the pull back of a free divisor $E \subset W$ by a map $i_0 : V \to W$ which is transverse to E outside 0. In the spirit of Damon's definition, we can also define our complex $\check{\Omega}^\bullet_{D_0}$ as the quotient of $\Omega^\bullet_V$ by the ideal generated by the pull-back via i_0 of a certain ideal of $\Omega^\bullet_W$. This is discussed in Section 3.

This programme, which of course can be applied to any category of spaces with a distinguished category of deformations, turns out to work extremely well for free and almost-free divisors. In fact if D is a free divisor then

$$\Omega^k_D/\text{torsion} = \Omega^k_X / h\Omega^k(\log D)$$

where h is an equation for D in X, and so, since all the modules $h\Omega^k(\log D)$ are free, the

$\check{\Omega}^k_D$ are maximal Cohen-Macaulay $\mathcal{O}_D$-modules. In Section 4 we show that as a consequence, the $\check{\Omega}^k_{D/S}$ enjoy precisely the same depth properties as the modules $\Omega^k_{X/S}$ on deformations of isolated complete intersection singularities (ICIS). In particular we prove an analogue of the fact, recalled above, that for an isolated hypersurface singularity the torsion has length τ:

Theorem 1.1 *Let $f_0 : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$, with $n \geq p$ nice dimensions, have finite $\mathcal{A}_e$-codimension, and let $D(f_0)$ be its discriminant. Then the torsion submodule of $\check{\Omega}^{p-1}_{D(f_0),0}$ has length equal to the $\mathcal{A}_e$ codimension of f_0.*

This is a consquence of 4.3(4) below.

In Sections 5, 6 and 7 below, we develop the theory of the Gauss-Manin connection on the vanishing cohomology of a deformation of an almost free divisor. The similarity with the classical theory — the Gauss-Manin connection on cohomology of the Milnor fibration of an isolated hypersurface singularity — is evident; many of the same theorems hold, including coherence, and with almost the same proofs. The author does not claim much originality in this latter part of the paper. However, what is remarkable is that the old constructions work so well in this new context.

This paper has taken a rather long time to write, in the course of which the author, and hopefully the paper, have profited from many helpful conversations with Francisco Castro and Luis Narváez, to whom I am very grateful. Both the idea of studying the Gauss-Manin connection on the discriminant of a family, and the definition of $\check{\Omega}^k_D$, arose in conversation with them. I am also grateful to Jim Damon and Duco van Straten for stimulating conversations on this topic, and to María Aparecida Ruas, Farid Tari and Washington Luiz Marar for an invitation to lecture on this topic in the 1998 São Carlos Singularities Workshop, which led to what I believe are some useful improvements in the presentation of this paper. I thank the referee for a very careful reading of the manuscript, and for a number of helpful suggestions.

Before the deadlines passed me by, I optimistically intended the paper for the proceedings of the 1996 Oberwolfach conference in honour of E.Brieskorn on his 60th birthday; notwithstanding the time that has since passed, the paper remains respectfully dedicated to him: imitation is the sincerest form of flattery

2 Almost-free divisors and their deformations

Let $E \subset W$ be a divisor in the manifold W, and let V be a manifold. The map $i : V \to W$ is *algebraically transverse* to E at $w = i(v)$ if $d_v i(T_v V) + T^{\log}_w E = T_w W$. Here $T^{\log}_w E$ is the "logarithmic" tangent space to E at w:

$$T^{\log}_w E = \{\chi(w) : \chi \in \mathrm{Der}(\log E)_w\}.$$

For future reference we note that the notion of logarithmic tangent space allows us to speak of logarithmic submersion and immersion, logarithmic critical value, etc. etc., as well as logarithmic (= algebraic) transversality. By 2.12 of [15] the logarithmic implicit function theorem holds (for maps with smooth target).

Definition 2.1 *([9]) The germ of hypersurface $D_0 \subset V, 0$ is an almost free divisor based on the germ of free divisor $E \subset W, 0$ if there is a map $i_0 : V, 0 \to W, 0$ which is algebraically transverse to E except at 0, such that $D_0 = i_0^{-1}(E)$.*

If the map i_0 of the definition is algebraically transverse to E then D_0 is a free divisor; in this case it is not too hard to show that there is a free divisor $E_0 \subset W_0$ and integers k and ℓ such that $(W, E) \simeq (W_0, E_0) \times \mathbb{C}^k$ and $(V, D_0) \simeq (W_0, E_0) \times \mathbb{C}^\ell$.

A 1-dimensional almost free divisor $D_0 \subset V$ based on the free divisor $E \subset W$ is also free in its own right, being a plane curve, even when the inducing map $i_0 : V \to W$ is not algebraically transverse to E. Although this phenomenon is not inherently problematic, it can be confusing, and therefore we now show that it cannot occur for higher dimensional almost free divisors.

Proposition 2.2 *Let $D_0 \subset V$ be an almost free divisor based on the free divisor $E \subset W$, $D_0 = i_0^{-1}(E)$. Then*
(1) The sequence

$$0 \to Der(\log D_0) \hookrightarrow \theta_V \xrightarrow{ti_0} \frac{\theta(i_0)}{i_0^*(Der(\log E))} \to \frac{\theta(i_0)}{T\mathcal{K}_{E,e} i_0} \to 0 \tag{1}$$

(where $T\mathcal{K}_{E,e} i_0 = ti_0(\theta_V) + i_0^*(Der(\log E))$) is exact.
(2) If i_0 is not algebraically transverse to E then depth $Der(\log D_0) = 2$ (we assume dim $V \geq 2$).

(3) (Jacobian criterion for freeness) If dim $V > 2$ then D_0 is a free divisor if and only if i_0 is algebraically transverse to E.

Proof First we prove that $i_0^*(\mathrm{Der}(\log E))$ is a free $\mathcal{O}_V$-module. Let $\xi_1, \cdots, \xi_n$ be a free basis for $\mathrm{Der}(\log E)$. We have an epimorphism

$$\oplus_1^n \mathcal{O}_V \xrightarrow{\phi} i_0^*(\mathrm{Der}(\log E))$$

with $\phi(e_j) = \xi_j \circ i_0$. Suppose $(a_1, \cdots, a_n) \in \ker\ \phi$. Then $\sum_j a_j(\xi_j \circ i_0) = 0$. However, if $i_0(x) \notin E$ then the vectors $\xi_1(i_0(x)), \cdots, \xi_n(i_0(x))$ are linearly independent, and so all $a_j(x)$ vanish. That is, $\mathrm{supp}(a_j) \subset D_0$. As D_0 is a divisor, all the a_j must be identically 0, and $i_0^*(\mathrm{Der}(\log E))$ is free.

(1) and (2) Consider the short exact sequence

$$0 \to i_0^*(\mathrm{Der}(\log E)) \to \theta(i_0) \to \theta(i_0)/i_0^*(\mathrm{Der}(\log E)) \to 0.$$

The right-most module is supported only on D_0 and thus is not free over $\mathcal{O}_V$, so its depth is less than the depth of $\theta(i_0)$, which is of course equal to dim V as $\theta(i_0)$ is a free $\mathcal{O}_V$-module. As $i_0^*(\mathrm{Der}(\log E))$ is also free, the depth lemma shows that depth $\theta(i_0)/i_0^*(\mathrm{Der}(\log E)) = \dim V - 1$.

The exact sequence

$$0 \to K \to \theta_V \xrightarrow{ti_0} \theta(i_0)/i_0^*(\mathrm{Der}(\log E)) \to \theta(i_0)/T\mathcal{K}_{E,e} i_0 \to 0$$

(where K and $T\mathcal{K}_{E,e} i_0$ are defined by the sequence) breaks into the two short exact sequences

$$0 \to \theta_V/K \to \theta(i_0)/i_0^*(\mathrm{Der}(\log E)) \to \theta(i_0)/T\mathcal{K}_{E,e} i_0 \to 0$$

and

$$0 \to K \to \theta_V \to \theta_V/K \to 0.$$

If $\theta(i_0)/T\mathcal{K}_{E,e}i_0 \neq 0$, then its depth is 0 (for it is supported only at 0). As depth $\theta(i_0)/i_0^*(\mathrm{Der}(\log E)) = \dim V - 1 > 0$, the depth lemma applied to the first of these two short exact sequences implies that depth $\theta_V/K = 1$. The second short exact sequence now implies that depth $K = 2$.

A similar argument shows that if $\theta(i_0)/T\mathcal{K}_{E,e}i_0 = 0$ then K must be free.

Now $K \subset \mathrm{Der}(\log D_0)$, for if $\chi \in K$ then for all $x \in D_0$, $di_0\,{}_x(\chi(x)) \in T_x^{\log}E$; if x is a regular point of D_0 this means that $i_0(x)$ is a regular point of E and that $\chi(x) \in (d_x i_0)^{-1}(T_{i_0(x)}E) = T_x D_0$, and thus $\chi \in \mathrm{Der}(\log D_0)$. Since i_0 is algebraically transverse to E outside 0, K and $\mathrm{Der}(\log D_0)$ coincide outside 0. Thus we have a short exact sequence

$$0 \to K \to \mathrm{Der}(\log D_0) \to \mathrm{Der}(\log D_0)/K \to 0$$

in which the right-most module, if not 0, has dimension 0 and thus depth 0. This is impossible, again by the depth lemma, since depth $K \geq 2$. Thus $K = \mathrm{Der}(\log D_0)$.

(3) "if" is clear; here we prove the converse. If $\theta(i_0)/T\mathcal{K}_{E,e}i_0 \neq 0$ then by (3) depth $\mathrm{Der}(\log D_0) = 2 < \dim V$, and thus D_0 cannot be a free divisor. $\square$

Example 2.3 The divisor D_0 shown in the Introduction is an almost free divisor. It is the preimage of the free divisor E consisting of the union of the four co-ordinate hyperplanes in $\mathbb{C}^4$, under the inducing map $i_0(x_1, x_2, x_3) = (x_1, x_2, x_3, x_1 + x_2 + x_3)$. Since i_0 is not algebraically transverse to E at 0 (for $T_0^{\log}E = \{0\}$), it follows from 2.2(3) that D_0 is not free at 0. Since D_0 is free outside 0, it also follows that i_0 is algebraically transverse to E outside 0 (though of course this is obvious anyway), so D_0 is an almost free divisor.

The deformations of almost free divisors that we consider arise as follows: beginning with an almost free divisor $D_0 \subset V$ obtained by pulling-back the free divisor $E \subset W$ by the map $i_0 : V \to W$, we consider deformations $i : V \times S \to W$ of i_0, and fibre $D := i^{-1}(E)$ over the base S of the deformation. When necessary we will refer to deformations of this type as *admissible*. Unless otherwise specified, *all deformations of almost free divisors that we consider from now on will be admissible*. A deformation $D \to S$ is *free* if $D = i^{-1}(E)$ with i algebraically transverse to E, is *versal* if i is a $\mathcal{K}_E$-versal deformation of i_0 (see 2.6 below), and *frees* D_0 if for generic $u \in S$ the map $i_u = i(\_, u)$ is algebraically transverse to E. The set of points $u \in S$ for which i_u is *not* algebraically transverse to E is the *logarithmic discriminant* of the deformation, and is denoted $\mathcal{B}$ (since in many cases it is the bifurcation set of a deformation). When the deformation is miniversal, the logarithmic discriminant is sometimes called the *$\mathcal{K}_E$-discriminant* of D_0 (or of i_0).

Theorem/Definition 2.4 *[12][9] Let D_0 be a p-dimensional almost free divisor based on the free divisor E. If $\pi : D \to S$ is a deformation of D_0 which frees D_0, then for $u \in S \setminus \mathcal{B}$, $D_s = i_u^{-1}(E)$ has the homotopy type of a wedge of p-spheres. The number of these is independent of the choice of deformation; it is called the* singular Milnor number *of D_0, and denoted $\mu_E(D_0)$. The space D_s is a* singular Milnor fibre *of D_0; up to homeomorphism it is independent of the choice of deformation.*

If the logarithmic version of Sard's theorem held for free divisors, then every free deformation of D_0 would free it. However, it does not hold:

Example 2.5 (F. J. Calderón, [5]) Let $D \subseteq \mathbb{C}^3$ be the set defined by the equation $xy(x-y)(x+\lambda y) = 0$, and let $\pi(x,y,\lambda) = \lambda$. One checks that D is a free divisor. The fibre of π over λ is a union of four lines through $(0,0)$ in $\mathbb{C}^2$. As λ varies, the cross-ratio of these four lines varies, and thus the family is not trivial. In particular, there is no logarithmic vector field with non-vanishing $\partial/\partial\lambda$-component at points $(0,0,\lambda)$, and so every point in $\mathbb{C}$ is a logarithmic critical value of π.

In [11] Jim Damon describes a large class of related examples; in particular the total space of the Hessian deformation of any non-simple weighted homogeneous plane curve singularity is a free divisor. Calderón's example falls in this class, of course.

It is probably sensible at this point to describe in more detail the family of examples we are most interested in. These are the discriminants of map-germs of isolated instability (i.e. finite $\mathcal{A}_e$ codimension) $f_0 : U = \mathbb{C}^n, 0 \to \mathbb{C}^p, 0 = V$ (with $n \geq p$), and those of their deformations which arise from deformations of the germ f_0. Any map $f_0 : U \to V$ can be obtained by transverse pull-back i_0 from a stable map F, as in the diagram

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{F} & \mathcal{V} \\ j_0 \uparrow & & \uparrow\ i_0 \\ U & \xrightarrow{f} & V \end{array}$$

In this case the discriminant $D(f_0)$ of f_0 is the pull-back by i_0 of the discriminant $D(F)$ of F. In [8], Damon shows that stability of f is equivalent to the algebraic transversality of i_0 to $D(F)$. We reprove this in Section 8 below. Thus, if f_0 has isolated instability then $D(f_0)$ is an almost free divisor based on $D(F)$.

Any unfolding $G : U\times S \to V\times S,\ g(x,s) = (g_s(x), s)$ of f_0 can also be obtained from F by transverse pull-back, by a deformation i of i_0, and thus the fibration $D(G) \to S$ is an admissible deformation of $D(f_0)$. Its fibre over $s \in S$ is the discriminant $D(g_s)$ of g_s. Now suppose that G is a stable unfolding of f_0, and that the logarithmic discriminant $\mathcal{B}$ of the projection $\pi : D(G) \to S$ is a proper subset of S (so that G is a stabilisation of f_0). For $u \notin \mathcal{B}$, f_u is a stable perturbation of f_0. The singular Milnor number $\mu_{D(F)}(D(f_0))$ of $D(f_0)$ is the rank of the vanishing homology of the discriminant of a stable perturbation of f_0; in [12] it is denoted by $\mu_\Delta(f_0)$ and called the discriminant Milnor number of f_0. The main theorem of [12] is that provided $(\dim U, \dim V)$ are in Mather's "nice dimensions", $\mu_\Delta(f_0))$ satisfies

$$\mu_\Delta(f_0) \geq \mathcal{A}_e\text{codimension of } f_0$$

with equality in case f_0 is weighted homogeneous. We give a fuller account of the link between the theory we construct here and the theory of right-left equivalence of map-germs, in Section 8 below. For now, we hope it will serve as a motivating example.

Now recall from [7] the notion of $\mathcal{K}_E$ equivalence for almost free divisors based on the free divisor E - or, more precisely, $\mathcal{K}_E$-equivalence of the maps inducing them.

Definition 2.6 *1. Let V and W be smooth, and let $E \subset W$ be a closed complex subspace. Then $\mathcal{K}_E$ is the subgroup of $\mathcal{K}$ consisting of pairs $(\Phi, \phi) \in Diff(V \times W) \times Diff(V)$ such that*

- $p_1(\Phi(v, w)) = \phi(v)$ *for* $(v, w) \in V \times W$*, where* $p_1 : V \times W \to V$ *is projection;*
- $\Phi(v, 0) = (\phi(v), 0)$ *for* $v \in V$*;*
- $\Phi(V \times E) = V \times E$.

2. Two germs $i_0, j_0 : V \to W$ are $\mathcal{K}_E$-equivalent if there exists $(\Phi, \phi) \in \mathcal{K}_E$ such that $\Phi \circ gr_{i_0} = gr_{j_0} \circ \phi$, where gr_{i_0} and gr_{j_0} are the graph embeddings of i_0 and j_0, $v \mapsto (v, i_0(v))$ and $v \mapsto (v, j_0(v))$ respectively.

Evidently if i_0 and j_0 are $\mathcal{K}_E$-equivalent then $i_0^{-1}(E)$ and $j_0^{-1}(E)$ are isomorphic varieties.

The $\mathcal{K}_{E,e}$ tangent space $T\mathcal{K}_{E,e} i_0$ is equal to

$$ti_0(\theta_V) + i_0^* \mathrm{Der}(logE);$$

the $\mathcal{K}_{E,e}$ normal space is then

$$\frac{\theta(i_0)}{ti_0(\theta_V) + i_0^* \mathrm{Der}(\log E)}.$$

By Nakayama's lemma, it is supported precisely on the points where i_0 fails to be algebraically transverse to E. The deformation i of i_0 is $\mathcal{K}_{E,e}$ versal if every other deformation of i_0 is parametrised $\mathcal{K}_E$ isomorphic, in an appropriate sense, to one induced from i. The sub-index "e" here indicates that in the parametrised equivalence, the parametrised family ϕ_s of diffeomorphisms of the domain V of $i + -$ are not required to fix $0 \in V$ when $s \neq 0$. Recquiring that $\phi_s(0) = 0$ for all $s \in S$ gives rise to a stricter notion of versality, namely $\mathcal{K}_E$-versality. Note that our terminology differs slightly from Damon's at this point: he does not consider the stricter notion, and uses the term $\mathcal{K}_E$-versal where we use $\mathcal{K}_{E,e}$ versal. See Section 3 of [33] for a discussion of this point.

Damon shows in [7] that the usual infinitesimal criterion for versality holds: i is $\mathcal{K}_{E,e}$-versal if and only if

$$T\mathcal{K}_{E,e} i_0 + \mathbb{C}\langle \dot{i}_1, \cdots, \dot{i}_d \rangle = \theta(i_0)$$

where $\dot{i}_j$ is the restriction to $V \times \{0\}$ of $\partial i / \partial s_j$, the $s_j, j = 1, \cdots, d$ being coordinates on S.

From the existence of $\mathcal{K}_{E,e}$-versal deformations it follows that if $\pi : D_1 \to S_1$ and $\pi_2 : D_2 \to S_2$ are both admissible deformations of the almost free divisor D_0 based on the free divisor E, and both free D_0, then the generic fibres of π_1 and π_2 are homeomorphic. For up to isomorphism both deformations are induced by base change from a versal deformation, and as the base space of the latter is smooth, and the discriminant is a proper analytic subvariety, it follows that any two generic fibres are homeomorphic.

Lemma 2.7 *Let $D_0 \subset V$ be an almost free divisor based on the free divisor $E \subset W$, with $i_0 : V \to W$ the inducing map. Let $i : V \times S$ be a deformation of i_0, algebraically transverse to E, let $D = i^{-1}(E)$, and let $\pi : D \to S$ be the resulting deformation of D_0. Then*

$$N\mathcal{K}_{E,e} i_0 \simeq \frac{\theta(\pi)}{t\pi(Der(\log D)) + \pi^*(m_{S,0})\theta(\pi)}.$$

Proof We have

$$\frac{\theta(\pi)}{t\pi(\mathrm{Der}(\log D)) + \pi^*(m_{S,0})\theta(\pi)} \simeq \frac{\theta_{V\times S}}{\theta_{V\times S/S} + \mathrm{Der}(\log D) + m_{S,0}\theta_{V\times S}}$$

$$\simeq \frac{\theta(j)}{tj(\theta_V) + j^*(\mathrm{Der}(\log D))} \;=\; N\mathcal{K}_{D,e}j,$$

where $j : V \to V \times S$ is the inclusion $v \mapsto (v, 0)$. Now $i_0 = i \circ j$, and because i is algebraically transverse to E, $ti : \theta(j) \to \theta(i_0)$ induces an isomorphism $N\mathcal{K}_{D,e}j \simeq N\mathcal{K}_{E,e}i$. This is proved in Proposition 1.5 of [8], but for the reader's convenience we now sketch the argument. It is easiest to see if we assume that $T_0^{\log}E = 0$; for then i must be a submersion, and choosing coordinates with respect to which it is a projection, the isomorphism is a straightforward calculation.

In the general case, if $T_0^{\log}E$ is k-dimensional, then the pair (W, E) is isomorphic in some neighbourhood to a product $(W_0, E_0) \times \mathbb{C}^k$. Let $p : W \to W_0$ be projection, using this product structure. Then the calculation just mentioned shows that $N\mathcal{K}_{E,e}i_0 \simeq N\mathcal{K}_{E_0,e}p \circ i_0$, and also that the submersion $p \circ i$ induces an isomorphism $N\mathcal{K}_{D,e}j \simeq N\mathcal{K}_{E_0,e}p \circ i_0$. □

In the light of this lemma, we refer to the quotient $\theta(\pi)/t\pi(\mathrm{Der}(\log D))$ as $T^{1,\log}_{D/S}$, and to $\theta(\pi)/t\pi(\mathrm{Der}(\log D)) + m_{S,0}\theta(\pi)$ as $T^{1,\log}_{D_0}$. The reader will recognise the similarity to the definition of $T^1_{X/S}$ and $T^1_{X_0}$ for a deformation $f : X \to S$ of an ICIS X_0, cf [22] Chapter 6.

We extend the definition of $T^{1,\log}_{D/S}$ to the case where $\pi : D \to S$ is not necessarily free:

Definition 2.8 *Let $\pi : D \to S$ be an admissible deformation, and let $\rho : \mathcal{D} \to S\times T$ be a free extension of π. Then*

$$T^{1,\log}_{D/S} = \frac{\theta(\rho)}{t\rho(Der(\log \mathcal{D})) + (t)\theta(\rho)},$$

where (t) is the ideal defining $S\times\{0\}$ in $S\times T$.

It is easily seen that this definition is independent of the choice of free extension ρ, by means of the following lemma, which is proved by standard singularity theory methods:

Lemma 2.9 *Let $E \subset W$ be a divisor, let $i_0 : V, 0 \to W, 0$, and suppose that $i : V\times T, (0,0) \to W, 0$ and $j : V \times T, (0,0) \to W, 0$ are deformations of i_0, both of them logarithmically transverse to E. Then i and j are $\mathcal{K}_E$-equivalent, as maps, by an equivalence which restricts to the identity on V.* □

The analogy with isolated complete intersection singularities continues:

Proposition 2.10 *Let $\pi : D \to S$ be an admissible deformation of the almost free divisor D_0, and suppose that π frees D_0. Then $T^{1,\log}_{D/S}$ is a Cohen-Macaulay module of dimension $\dim S - 1$.*

Proof If π is a free deformation, then the result follows from a theorem of Buchsbaum and Rim ([3] Corollary 2.7); for we have an exact sequence

$$\mathrm{Der}(\log D) \to \theta(\tilde{\pi}) \to T^{1,\log}_{D/S} \to 0$$

(where $\tilde{\pi} : V \times S \to S$ is the obvious extension of π) in which $\mathrm{Der}(\log D(F))$ is free of rank $\dim D + 1$ and $\theta(\pi)$ is free of rank $\dim S$. The theorem cited implies that $\dim T^{1,\log}_{D/S} \geq \dim S - 1$, and that if this dimension is attained then $T^{1,\log}_{D/S}$ is Cohen-Macaulay. Now the hypothesis that D_0 be almost free implies that $T^{1,\log}_{D/S}$ is finite over S, while the hypothesis that π frees D_0 implies that $\dim(\mathrm{supp}(\pi_*(T^{1,\log}_{D/S})) < \dim S$. Hence $\dim T^{1,\log}_{D/S} = \dim S - 1$, and (1) follows.

The general case follows from this special case by a standard dimensional argument. Let $\rho : \mathcal{D} \to S \times T$ be a free extension of π, with $\dim T = d$. Then because

$$T^{1,\log}_{D/S} = (T^{1,\log}_{\mathcal{D}/S\times T})/(t_1, \cdots, t_d)T^{1,\log}_{\mathcal{D}/S\times T}$$

and $\dim T^{1,\log}_{D/S} = \dim\,(T^{1,\log}_{\mathcal{D}/S\times T}) - d$, $T^{1,\log}_{D/S}$ also is Cohen-Macaulay.

□

3 Differential forms on free and almost free divisors

From now on, when we speak of a free divisor the letter h will always denote its reduced equation.

In this section we define and study the improved version of Kähler forms on free divisors mentioned in the introduction. First we recall (e.g. from [22]) the standard defintion of the $\mathcal{O}_X$-module Ω^k_X of Kähler k-forms on a subvariety X of a smooth space V. Suppose that the ideal of functions vanishing on X is generated by $f_1, \ldots, f_n$; then Ω^k_X is defined to be

$$\frac{\Omega^k_V}{(f_1, \ldots, f_n)\Omega^k_V + \sum_i df_i \wedge \Omega^{k-1}_V}.$$

The exterior derivative $d : \Omega^k_V \to \Omega^{k+1}_V$ passes to the quotient to give an exterior derivative $\Omega^k_X \to \Omega^{k+1}_X$; the complex $\Omega^\bullet_X$, equipped with this exterior derivative, is naturally isomorphic to the complex of holomorphic differential forms on X defined using charts, when X is smooth.

Now let $D \subset V$ be a free divisor. Recall K.Saito's definition in [31] of the sheaf $\Omega^k(\log D)$ as the $\mathcal{O}_V$-module of meromorphic differential forms ω on V such that $h\omega$ and $h\, d\omega$ are regular. Clearly $\Omega^\bullet_V(\log D)$ is a complex; note that $h\Omega^\bullet_V(\log D)$ is a subcomplex of $\Omega^\bullet_V$, so that the quotient $\Omega^\bullet_V/h\Omega^\bullet_V(\log D)$ is also a complex. This complex is our replacement for the complex of Kähler forms on D.

Definition 3.1 *Let D be a free divisor. We define*

$$\check{\Omega}^k_D = \frac{\Omega^k_W}{h\Omega^k(\log D)}.$$

Since $(dh)/h \wedge \Omega^{k-1}_V + \Omega^k_V \subseteq \Omega^k_V(\log D)$, $\check{\Omega}^k_D$ is a quotient of Ω^k_D.

Example 3.2 Let D be the normal crossing divisor $z_1 \ldots z_p = 0$. Then $\Omega^1_V(\log D)$ is the free $\mathcal{O}_V$-exterior algebra on generators $dz_i/z_i, i = 1, \cdots, p$, and so $h\Omega^k_V(\log D)$ is generated

over $\mathcal{O}_V$ by the forms $z_J dz_I$, where $|I| = k$, $J \cup I = \{1, \cdots, p\}$ and $J \cap I = \emptyset$. It is easy to see that if $n : \bar{D} \to D$ is the normalisation of D, and $i : D \to \mathbb{C}^{n+1}$ is inclusion, then $h\Omega^k_V(\log D) = \ker(i \circ n)^* : \Omega^k_V \to \Omega^k_{\bar{D}}$, and so $\check{\Omega}^k_D \hookrightarrow n_* \Omega^k_{\bar{D}}$. By contrast, Ω^k_D is larger, and the kernel of $n^* : \Omega^k_D \to \Omega^k_{\bar{D}}$ is non-trivial: for example, the forms $(h/x_i)dx_i$ are not equal to 0 in Ω^1_D.

Lemma 3.3 *Let D be a divisor in the $p+1$-dimensional complex manifold V, with local defining equation h. Then*

(1) $\check{\Omega}^0_D = \mathcal{O}_D$.

(2) $\check{\Omega}^{p+1}_D = 0$.

(3) For k with $0 \le k \le p$, the support of $\check{\Omega}^k_D$ is equal to D.

(4) If x is a smooth point of D, then $\check{\Omega}^k_{D,x} = \Omega^k_{D,x}$.

(5) If D is a free divisor, then the depth (as module over $\mathcal{O}_V$ and $\mathcal{O}_D$) of $\check{\Omega}^k_D$ is equal to dim$D$, for $0 \le k \le p$, (so that $\check{\Omega}^k_D$ is a maximal Cohen Macaulay $\mathcal{O}_D$-module).

(6) If the germ of D at x is quasihomogeneous, then the complex

$$0 \to \mathbb{C} \to \mathcal{O}_D = \check{\Omega}^0_D \to \check{\Omega}^1_D \to \cdots \to \check{\Omega}^p_D \to 0$$

is exact at x.

Proof (1) and (3) are obvious.

(2) follows from the fact that $\Omega^{p+1}(\log D) = (1/h)\Omega^{p+1}_V$.

(4). If y is a smooth point of D, then $\Omega^k(\log D)_y = (dh/h) \wedge \Omega^{k-1}_{V,y} + \Omega^k_{V,y}$. Hence $h\Omega^k(\log D) = dh \wedge \Omega^{k-1}_{V,y} + h\Omega^k_{V,y}$, and $\Omega^k_{V,y}/h\Omega^k(\log D) = \Omega^k_{D,y}$.

(5). $\Omega^k_V/h\Omega^k_V(\log D)$ has an $\mathcal{O}_V$-free resolution $0 \to h\Omega^k(\log D) \to \Omega^k_V \to \Omega^k_V/h\Omega^k(\log D) \to 0$, so its projective dimension is less than or equal to 1. Since its support is D, it must have projective dimension 1 and thus depth $p-1$.

(6). The proof here is a variant of the elementary proof that the usual complex

$$0 \to \mathbb{C} \to \mathcal{O}_D \to \Omega^1_D \to \cdots \to \Omega^{p-1}_D \to \Omega^p_D \to 0$$

is exact, which can be found, for example, as Lemma 9.9 in [22]. We use the local $\mathbb{C}^*$-action centred at y. Let χ_e be a local Euler field, vanishing at y. Then for any homogeneous form $\omega \in \Omega^k_{V,y}$ of weight ℓ, we have $L_{\chi_e}(\omega) = \ell\omega$. Note Cartan's identity $L_{\chi_e}(\omega) = \iota_{\chi_e}(d\omega) + d\iota_{\chi_e}(\omega)$. Now suppose that $d\omega \in h\Omega^k(\log D)$, and for brevity write $d\omega = h\sigma$. Then $\ell\omega = h\iota_{\chi_e}(\sigma) +$

$d\iota_{\chi_e}(\omega) \in h\Omega^k(\log D)_y + d\Omega^{k-1}_{V,y}$. Thus, unless $\ell = 0$, the class of ω in $\Omega^k_{V,y}/h\Omega^k(\log D)_y$ is a coboundary. It follows that if we define $H : \Omega^\bullet_{V,y}/\Omega^\bullet(\log D) \to \Omega^{\bullet-1}_{V,y}/h\Omega^{\bullet-1}(\log D)_y$ to be $(1/\ell)\iota_{\chi_e}$ on the weight ℓ subspace (and extend continuously to the completion), then H gives a homotopy equivalence between $\Omega^\bullet_{V,y}/\Omega^\bullet(\log D)_y$ and its weight zero subspace. This weight 0 subspace is just $\mathbb{C}$ in degree 0, since our $\mathbb{C}^*$ action is good (i.e. the weights of all the variables are positive). □

Corollary 3.4 *Let $D \subset V$ be a free divisor. For each k, with $0 \leq k \leq \dim V$, the torsion submodule of Ω^k_D is equal to $h\Omega^k(\log D)/h\Omega^k_V + dh \wedge \Omega^{k-1}_V$, and thus $\check{\Omega}^k_D = \Omega^k_D/$torsion.*

Proof As $\check{\Omega}^k_D$ is Cohen-Macaulay it is torsion free, and thus

$$T\Omega^k_D \subseteq h\Omega^k(\log D)/h\Omega^k_V + dh \wedge \Omega^{k-1}_V.$$

However the right hand side here is torsion, as it is supported only on D_{sing} (cf. 3.3.5.) □

The fact that on a free divisor $\check{\Omega}^\bullet_D$ is a complex of maximal Cohen-Macaulay modules on D is crucial to its applications. In this respect it improves on the usual complex $\Omega^\bullet_D$; in fact

Proposition 3.5 *If D is a free divisor and singular at x then for $1 \leq k \leq \dim D$, depth $\Omega^k_D = \dim D - 1$.*

Proof This follows by applying the depth lemma to each of the three short exact sequences

$$0 \to \Omega^k_V \to \frac{\Omega^k(\log D)}{\Omega^k_V} \to 0,$$

$$0 \to \frac{\Omega^{k-1}_V}{h\Omega^{k-1}(\log D)} \xrightarrow{(dh/h)\wedge} \frac{\Omega^k(\log D)}{\Omega^k} \to \frac{\Omega^k(\log D)}{(dh/h) \wedge \Omega^{k-1} + \Omega^k} \to 0$$

and

$$0 \to \frac{\Omega^k(\log D)}{(dh/h) \wedge \Omega^{k-1} + \Omega^k} \xrightarrow{\times h} \frac{\Omega^k}{dh \wedge \Omega^{k-1} + h\Omega^k} \to \frac{\Omega^k}{h\Omega^k(\log D)} \to 0$$

together with 3.6 below. For the first shows that $\Omega^k(\log D)/\Omega^k_V$ has depth $p := \dim D$, and so the second shows that its right-hand member has depth $p-1$ (since by 3.3(4) is is supported only on D_{sing}. and by 3.6 below it is not 0). The third sequence has Ω^k_D as its middle term; the depth lemma says that its depth is at least as great as the minumum of the depths of the two outer terms, and thus at least $p-1$. It also says that if it is greater than this minumum then the first term has depth one more than the last. Since the first term has depth less than the last, this cannot happen. □

Lemma 3.6 *If x is a singular point on the reduced free divisor $D \subseteq \mathbb{C}^{p+1}$ then for $0 < k \leq p+1$, $dh \wedge \Omega^{k-1}_{\mathbb{C}^{n+1},x} + h\Omega^k_{\mathbb{C}^{n+1},x}$ is strictly contained in $h\Omega^k(\log D)_x$.*

Proof Suppose $\Omega^1(\log D)_x = \Omega^1_{\mathbb{C}^{p+1},x} + (dh/h) \wedge \Omega^0_{\mathbb{C}^{p+1},x}$. Then a free basis for $\Omega^1(\log D)_x$ can be extracted from the list $dx_1, \cdots, dx_{p+1}, dh/h$. If $\omega_1, \cdots, \omega_{p+1}$ is a free basis, then $\omega_1 \wedge \cdots \wedge \omega_{p+1} = dx_1 \wedge \cdots \wedge dx_{p+1}/h$, up to multiplication by a unit ([31]). Thus, dh/h must be a member of any free basis extracted from this list, and so without loss of generality, we can suppose that $dx_1, \cdots, dx_p, dh/h$ is a free basis. Since dx_{p+1} is then an $\mathcal{O}_{\mathbb{C}^{p+1}}$-linear combination of $dx_1, \cdots, dx_p, dh/h$, a calculation shows that $\partial h/\partial x_{p+1}$ is a unit, and thus that D is non-singular at x.

Let $\omega_1, \ldots, \omega_{p+1}$ be a free basis for $\Omega^1(\log D)$. Then $\Omega^\bullet(\log D)_x$ is the free $\mathcal{O}_{\mathbb{C}^{p+1},x}$ exterior algebra over $\mathcal{O}_{\mathbb{C}^{p+1},x}$ generated by $\omega_1, \ldots, \omega_{p+1}$, while the algebra $\Omega^\bullet_{\mathbb{C}^{p+1},x} + (dh/h) \wedge \Omega^{\bullet-1}_{\mathbb{C}^{p+1},x}$ is a quotient (because there are relations in general) of the free $\mathcal{O}_{\mathbb{C}^{p+1},x}$ exterior algebra generated by $dx_1, \ldots, dx_{p+1}, dh/h$. Thus, if $\Omega^1_{\mathbb{C}^{p+1},x} + (dh/h) \wedge \Omega^0_{\mathbb{C}^{p+1},x}$ is strictly contained in $\Omega^1(\log D)_x$ then for each k with $0 < k \leq p+1$, $\Omega^k_{\mathbb{C}^{p+1},x} + (dh/h) \wedge \Omega^{k-1}_{\mathbb{C}^{p+1},x}$ is strictly contained in $\Omega^k(\log D)_x$. □

Recall that $\theta_D = \mathrm{Hom}_{\mathcal{O}_D}(\Omega^1_D, \mathcal{O}_D)$ is equal to the restriction to D of $\mathrm{Der}(\log D)$. The next proposition characterises $\check{\Omega}^1_D$ as the double dual of Ω^1_D.

Proposition 3.7 *Let $D \subset V$ be a free divisor; then evaluation of forms on vector fields induces a perfect pairing $\check{\Omega}^1_D \times \theta_D \longrightarrow \mathcal{O}_D$.*

Proof The result is well known when D is non-singular, so we assume D is singular. Define N by the short exact sequence $0 \to \mathrm{Der}(\log D) \to \theta_V \to N \to 0$. It is a maximal Cohen-Macaulay $\mathcal{O}_D$ module. Because $\mathrm{Der}(\log D) \supset h\theta_V$, restricting to D gives the exact sequence $0 \to \theta_D \to \theta_V|_D \to N \to 0$, and now dualising with respect to $\mathcal{O}_D$ gives an exact sequence

$$0 \to N^\vee \to (\theta_V|_D)^\vee \to \theta_D^\vee \to \mathrm{Ext}^1_D(N, \mathcal{O}_D) \to 0.$$

Because N is a maximal Cohen-Macaulay module on a singular hypersurface, it has a 2-periodic $\mathcal{O}_D$-free resolution (see section 6 of [14]), and this remains exact on dualising with respect to $\mathcal{O}_D$; hence $\mathrm{Ext}^1_D(N, \mathcal{O}_D) = 0$. Thus $\Omega^1_V|_D$ maps onto $\theta^\vee_D$, and hence so does Ω^1_V. The kernel of the epimorphism $\Omega^1_V \to \theta^\vee_D$ is $\{\omega \in \Omega^1_V : \omega(\chi) \in (h)\ \forall \chi \in \mathrm{Der}(\log D)\}$. Denote this by K. Since evaluation of forms on vector fields gives an isomorphism $\Omega^1(\log D) \simeq \mathrm{Hom}_V(\mathrm{Der}(\log D), \mathcal{O}_V)$, K is equal to $h\Omega^1(\log D)$. Thus, $(\theta_D)^\vee$ is equal to $\check{\Omega}^1_D$. Finally, since $\check{\Omega}^1_D = (\Omega^1_D)^{\vee\vee}$, its dual must be $(\Omega^1_D)^\vee$, i.e. θ_D. □

We now define differential forms on almost free divisors. Rather than beginning with the procedure outlined in the introduction, we give an alternative definition, more in the spirit of Damons's definition of almost free divisor, in terms of the inducing map i_0. The two approaches are shown to be equivalent in 3.9 below.

Definition 3.8 *Let $E \subset W$ be a free divisor, let $i_0 : V \to W$ be a map and let $i : V \times S \to W$ a deformation of i_0. Let $D_0 = i_0^{-1}(E)$ and $D = i^{-1}(E)$. We set*

1.

$$\check{\Omega}^\bullet_{D_0} = \frac{\Omega^\bullet_V}{\langle i_0^*(h\Omega^\bullet(\log E))\rangle}$$

2.

$$\check{\Omega}^\bullet_{D/S}=\frac{\Omega^\bullet_{V\times S}}{\langle i^*(h\Omega^\bullet(\log E))\rangle+\sum ds_i\wedge\Omega^{\bullet-1}_{V\times S}}.$$

Here $\langle i_0^*(h\Omega^\bullet(\log E))\rangle$ and $\langle i^*(h\Omega^\bullet(\log E))\rangle$ are the ideals in the exterior algebras $\Omega^\bullet_V$ and $\Omega^\bullet_{V\times S}$ generated by the forms $i_0^*(\omega)$ and $i^*(\omega)$, for $\omega\in h\Omega^\bullet(\log E)$. They are of course subcomplexes.

Proposition 3.9 *Suppose that $i: V\times S\to W$ is a deformation of $i_0: V\to W$. Let h_E be a reduced equation for E, so that $h_D=h\circ i$ is a reduced equation for $D=i^{-1}(E)$. Then*

1.

$$\check{\Omega}^k_{D_0}=\frac{\check{\Omega}^k_{D/S}}{\sum s_i\check{\Omega}^k_{D\times S}}$$

(so the construction of $\check{\Omega}^\bullet$ commutes with restriction).

2. If I is algebraically transverse to E then

$$\check{\Omega}^k_{D/S}=\frac{\Omega^k_{V\times S}}{h_D\Omega^k(\log D)+\sum_i ds_i\wedge\Omega^{k-1}_{V\times S}}$$

Proof The first statement is obvious. For the second, we need a lemma.

Lemma 3.10 *Let $p: U\to W$ be a submersion, let $E\subset W$ be a divisor and let $D=p^{-1}(E)$. Then*

$$\Omega^\bullet(\log D)=\langle p^*(\Omega^\bullet(\log E))\rangle.$$

Proof Inclusion of the right hand side in the left follows from the obvious fact that $p^*(\Omega^\bullet(\log E))\subset\Omega^\bullet(\log D)$. To prove the opposite inclusion note that after a diffeomorphism we may assume $U=W\times T$ and $D=E\times T$, and that $p: U\to W$ is just the projection $W\times T\to W$. Let $q: W\to W\times T$ be the inclusion $w\mapsto(w,0)$, and let $t_1,\cdots,t_n$ be coordinates on T; then if $\omega\in\Omega^\ell(\log D)$, we have

$$\begin{aligned}\omega(x,t)-\omega(x,0)&=\int_0^1\tfrac{d}{ds}\omega(x,st)ds\\&=\int_0^1\sum_i t_i\tfrac{\partial}{\partial t_i}\omega(x,st)ds\\&=\sum_i t_i\int_0^1\tfrac{\partial}{\partial t_i}\omega(x,st)ds.\end{aligned}$$

Now each form $\int_0^1\frac{\partial}{\partial t_i}\omega(x,st)ds$ is in $\Omega^\ell(\log D)$, for taking as equation for D the equation h of E (regarded as a function of (x,t) independent of t) we have

$$h(x)\int_0^1\frac{\partial}{\partial t_i}\omega(x,st)ds=\int_0^1\frac{\partial}{\partial t_i}h(x)\omega(x,st)ds$$

and is therefore regular, and similarly

$$dh\wedge\int_0^1\frac{\partial}{\partial t_i}\omega(x,st)ds=\int_0^1\frac{\partial}{\partial t_i}dh\wedge\omega(x,st)ds,$$

also regular.

Hence

$$\omega \equiv \omega(x,0) \pmod{m_{U,0}\Omega^\ell(\log D)}$$

and the required inclusion will follow by Nakayama's Lemma, once we show that

$$\omega(x,0) \in p^*(\Omega^\bullet(\log E)) \wedge \Omega^\bullet_U.$$

But this is now evident: if

$$\omega(x,0) = \sum dt_{i_1} \wedge \cdots \wedge dt_{i_k} \wedge \omega_{i_1,\cdots,i_k}$$

with each $\omega_{i_1,\cdots,i_k}$ independent of t and having no dt_i component, then each $\omega_{i_1,\cdots,i_k}$ is D-logarithmic (as it is the contraction of $\omega(x,0)$ by a D-logarithmic k-vector) and, since $\omega_{i_1,\cdots,i_k} = p^*q^*\omega_{i_1,\cdots,i_k}$, we have $\omega_{i_1,\cdots,i_k} \in p^*(\Omega^\bullet(\log E))$. □

Part (2) of the proposition now follows: we have

$$\check{\Omega}^\bullet_{D/S} = \frac{\check{\Omega}^\bullet_D}{\sum ds_i \wedge \check{\Omega}^{\bullet-1}_D} = \frac{\Omega^\bullet_{V\times S}}{\langle i^*(h_E\Omega^\bullet(\log E))\rangle + \sum ds_i \wedge \Omega^{\bullet-1}_{V\times S}} = \frac{\Omega^\bullet_{V\times S}}{h_D\Omega^\bullet(\log D) + \sum ds_i \wedge \Omega^{\bullet-1}_{V\times S}}.$$

□

Proposition 3.11 *If $E \subset W$ is a free divisor, $i_0 : V \to W$ is a map and $D_0 = i_0^{-1}(E)$, then we have the following inclusions:*

$$h_{D_0}\Omega^k_V + dh_{D_0} \wedge \Omega^{k-1}_V \subseteq \langle i_0^*(h_E\Omega^\bullet(\log E)\rangle_k \subseteq h_{D_0}\Omega^k(\log D_0)$$

where the lower index k in the middle complex means the degree k summand. If D_0 is almost free, then provided dim $D_0 \geq 2$ and E is not smooth, the first inclusion is strict. If i_0 is not algebraically transverse to E then the second inclusion is also strict for $k =$ dim D_0 and $k =$ dim $D_0 + 1$.

Proof The first inclusion holds because $dh_E/h_E \in \Omega^1(\log E)$ and we can take $h_{D_0} = i_0^*(h_E)$; thus $h_{D_0} \in i_0^*(h_E\Omega^0(\log E))$ and $dh_{D_0} \in i_0^*(h_E\Omega^1(\log E))$. The second inclusion holds because, as is easily checked directly from the definition, $i_0^*(\Omega^\bullet(\log E)) \subset \Omega^\bullet(\log D_0)$.

Strictness of the first inclusion when D_0 is almost free holds by 3.6; for this implies that the quotient of the right hand side by the left is supported at all points of $D_{0,\text{sing}} \setminus \{0\}$, and thus also at 0.

Strictness of the second inclusion for $k =$ dim D_0 follows from the fact that the torsion submodule of $\check{\Omega}^p_{D_0}$ has length equal to that of $N\mathcal{K}_{E,e}i$ (by 4.3, below), and thus depth $\check{\Omega}^p_{D_0} = 0$. The depth of $\Omega^p_V/h_{D_0}\Omega^p(\log D_0)$, on the other hand, is at least 1; for $\Omega^p(\log D_0)$, and thus $h_{D_0}\Omega^p(\log D_0)$, are isomorphic to $\mathrm{Der}(\log D_0)$ and thus have depth at least 2, from which it follows that $\Omega^p_V/h_{D_0}\Omega^p(\log D_0)$ has depth at least 1.

For $k =$ dim V, it is evident that $i_0^*(h_E\Omega^k(\log E)) \subseteq m_{V,0}\Omega^k_V$ unless i_0 is algebraically transverse to E. □

Proposition 3.12 *Let D_0 be an almost free divisor. Then for each vector field $\chi \in Der(\log D_0)$, contraction by χ gives rise to a well-defined $\mathcal{O}_{D_0}$-linear morphism $\check{\Omega}^k_{D_0} \to \check{\Omega}^{k-1}_{D_0}$.*

Proof The content of the statement is that if we represent $\omega \in \check{\Omega}^k_{D_0}$ by $\omega_1 \in \Omega^k_V$ then the class of $\iota_\chi(\omega_1)$ in $\check{\Omega}^{k-1}_{D_0}$ is independent of the choice of ω_1. Since contraction is evidently linear, this amounts to showing that if $\omega_1 \in \Omega^k_V$ is 0 in $\check{\Omega}^k_{D_0}$ in then $\iota_\chi(\omega_1)$ is 0 in $\check{\Omega}^{k-1}_{D_0}$.

Let $i : V \times S \to W$ be a deformation of the inducing map i_0 which is algebraically transverse to E, and let $D = i^{-1}(E)$. Denote by $j : V \to V \times S$ the inclusion $j(v) = (v, 0)$. We have $D_0 = j^{-1}(D)$, and since D is free we can apply Proposition 2.2 to conclude that $\mathrm{Der}(\log D_0) = tj^{-1}(j^*(\mathrm{Der}(\log D)))$. As j is an inclusion this means simply that for each $\chi \in \mathrm{Der}(\log D_0)$ there exists $\chi_2 \in \mathrm{Der}(\log D)$ extending χ. In particular, if $\chi_2 = \sum a_i\partial/\partial v_i + \sum b_j\partial/\partial s_j$ then all b_j vanish when $s = 0$.

Let $\omega_2 \in \Omega^k_{V\times S}$ extend ω_1. As ω_1 is equal to 0 in $\check{\Omega}^k_{D_0}$, we have

$$\omega_2 \in h\Omega^k(\log D) + \sum s_i\Omega^k_{V\times S} + \sum ds_i \wedge \Omega^{k-1}_{V\times S}.$$

Now let $\chi_2 \in \mathrm{Der}(\log D)$ extend χ. Then $\iota_\chi(\omega_1) = j^*(\iota_{\chi_2}(\omega_2))$. Since

$$\iota_{\chi_2}(h\Omega^k(\log D)) \subset h\Omega^{k-1}(\log D),$$

$$\iota_{\chi_2}(\sum s_i\Omega^k_{V\times S}) \subset \sum s_i\Omega^{k-1}_{V\times S}$$

and

$$\iota_{\chi_2}(\sum ds_i \wedge \Omega^k_{V\times S}) \subset \sum ds_i \wedge \Omega^{k-2}_{V\times S} + \sum s_i\Omega^{k-1}_{V\times S},$$

it follows that $\iota_\chi(\omega_1) = j^*(\iota_{\chi_2}(\omega_2) \in j^*(h\Omega^{k-1}(\log D) + \sum s_i\Omega^{k-1}_{V\times S} + \sum ds_i \wedge \Omega^{k-2}_{V\times S})$ and the proposition is proved. □

We end this section with a definition of logarithmic critical space:

Definition 3.13 *Let $\pi : D \to S$ be a deformation of an almost free divisor and choose a free extension $\rho : \mathcal{D} \to S \times T$ of π. Then*

1. *the* logarithmic critical space *of ρ, $C^{\log}_\rho$, is*

$$C^{\log}_\rho = supp T^{1,\log}_{\mathcal{D}/S\times T}$$

with analytic structure defined by the ideal $\mathcal{C}^{\log}_\rho := \mathcal{F}_0^{\mathcal{O}_{V\times S\times T}}(T^{1,\log}_{\mathcal{D}/S\times T})$

2. *the logarithmic critical space of π is $C^{\log}_\pi = C^{\log}_\rho \cap \rho^{-1}(S \times \{0\})$, with its natural analytic structure as an intersection.(Once again, this does not depend on the choice of free extension ρ of π.)*

Here $\mathcal{F}_0$ is the zero'th Fitting ideal. We remark that in case the codimension of $C^{\log}_p$ in $V \times S \times T$ is equal to $\dim S \times T + 1$, then this coincides with the annihilator of the $\mathcal{O}_{V\times S\times T}$-module $\theta(\pi)/t\pi(\mathrm{Der}(\log D)$, by the theorem of [4]. This is the case, for example, if $\pi : D \to S$ is a free deformation of an almost free divisor D_0, and frees D_0. The ideal $\mathcal{C}^{\log}_\pi$ can be calculated very

easily: if Θ is a matrix whose columns are the components of a free basis of $\mathrm{Der}(\log \mathcal{D})$ and Θ' its submatrix consisting of the rows corresponding to the deformation parameters, then $\mathcal{C}_\rho^{\log}$ is generated by the restriction to $V\times S$ of the maximal minors of Θ'.

One of the major differences with the classical case is that $\mathcal{C}_\pi^{\log}$ is not necessarily radical even when $\pi : D \to S$ is the deformation induced by a $\mathcal{K}_{E,e}$-versal deformation of the inducing map i (cf [10]).

4 Algebraic properties

As remarked above, our complex $\check{\Omega}^\bullet_{D/S}$ has better depth properties than the standard complex $\Omega^\bullet_{D/S}$. The improvement is crucial.

Lemma 4.1 *Let $D \subseteq \mathbb{C}^N$ be a free divisor, and let $g_1, \cdots, g_s$ and $f_1, \cdots, f_m$ be holomorphic functions on D, such that $g_1, \cdots, g_s$ is a regular sequence on $\mathcal{O}_D$, and such that for each i, dim $C_g^{\log} \cap V(g_1, \cdots, g_i) <$ dim $D \cap V(g_1, \cdots, g_i)$ (where $C_g^{\log}$ is the intersection with D of the support of $\theta(g)/tg(Der(\log G))$). Denote by f the map $(f_1, \cdots, f_m) : D \to \mathbb{C}^m$. Let $\mathcal{O} = \mathcal{O}_D/(g_1, \cdots, g_s)$ and*

$$\check{\Omega}^k := \check{\Omega}^k_D/(g_1, \cdots, g_s)\check{\Omega}^k_D.$$

$$\check{\Omega}^k(i) = \check{\Omega}^k/\sum_{j=1}^{i} df_j \wedge \check{\Omega}^{k-1}.$$

Then

(1) for $0 \le i \le m$, and $0 \le k <$ dim $V(g) -$ dim $C_f^{\log} \cap V(g)$, depth $\check{\Omega}^k_x(i) \ge$ dim$V(g) - k$, and

(2) the map

$$df = df_1 \wedge \cdots \wedge df_m : \frac{\check{\Omega}^k_{D,x}}{(g_1, \cdots, g_s)\check{\Omega}^k_{D,x} + \sum_{j=1}^s df_j \wedge \ \check{\Omega}^{k-1}_{D,x}} \to \check{\Omega}^{k+m}_x$$

is injective, for $0 \le k <$ depth $\mathcal{O}_D/(g_1, \cdots, g_s) -$ dim$_x C_f^{\log} \cap V(g)$.

Proof This lemma is just a translation of Lemma 1.6 of [19] (which can also be found in section 8C of [22]) and the proof is essentially the same. First, note that because $\check{\Omega}^k_D$ is a maximal Cohen-macaulay $\mathcal{O}_D$-module, any $\mathcal{O}_D$-regular sequence is also $\check{\Omega}^k_D$-regular, and thus $\check{\Omega}^k$ is a maximal Cohen-Macaulay $\mathcal{O}$-module. We now follow Greuel's argument. Write $f(x) = z$, and $f^{-1}(z) = D_z$.

(a) If $x \notin C_f^{\log} \cap V(g)$ then D, x is isomorphic over $\mathbb{C}^s, z$ to $D_z \times \mathbb{C}^s$. The map

$$df_1 \wedge \cdots \wedge df_s \wedge : \check{\Omega}^k_{D/\mathbb{C}^s} \to \check{\Omega}^{k+s}_D$$

then has a left inverse, and it follows that it remains exact after tensoring with $\mathcal{O}_x$.

(b) Suppose that $x \in C_f^{\log} \cap V(g)$.

We prove (1) by induction on k. For $k = 0$ it is trivial. Suppose it true for $k-1$, and consider the sequence

$$0 \to \check{\Omega}_x^{k-1}(i) \xrightarrow{df_i\wedge} \check{\Omega}_x^k(i-1) \to \check{\Omega}_x^k(i) \to 0,$$

which is evidently exact except perhaps at $\check{\Omega}_x^{k-1}(i)$. To see that $df_i\wedge$ is injective, note that it is certainly injective off $C_f^{\log} \cap V(g)$, and thus $\ker df_i\wedge \subseteq \mathcal{H}^0_{C_f^{\log}\cap V(g)}(\check{\Omega}_x^{k-1}(i))$. However, $\mathrm{depth}\check{\Omega}_x^{k-1}(i) > \mathrm{depth}\mathcal{O}_x - k > \mathrm{dim}C_f^{\log} \cap V(g)$, and so this local cohomology group vanishes and the sequence is exact. Now we argue by induction on i: for $i = 0$, depth $\check{\Omega}_x^k(i) =$ depth $\check{\Omega}_x^k =$ depth $\mathcal{O}_x$, as shown at the start of this proof. Suppose that depth $\check{\Omega}_x^k(i-1) \geq$ depth $\mathcal{O}_x - k$. Then by exactness of the above sequence, it follows that depth $\check{\Omega}_x^k(i) \geq$ depth $\mathcal{O}_x - k$, as required. This completes the proof of (1).

Part (2) follows, for now by (a), $\ker df\wedge : \check{\Omega}_x^k/\sum_{j=1}^m df_j \wedge \check{\Omega}_x^{k-1} \to \check{\Omega}_x^{k+m}$ is contained in $\mathcal{H}^0_{C_f^{\log}\cap V(g)}(\check{\Omega}_x^k/\sum_{j=1}^m df_j \wedge \check{\Omega}_x^{k-1})$, which, by the depth estimate just obtained, is equal to 0. □

We now apply the lemma in the following situation: $\pi : D \to S$ is an admissible deformation of an almost free divisor, $\rho : \mathcal{D} \to S\times T$ is a free extension of π, and denoting coordinates on S and T by $s_1, \cdots, s_d$ and $t_1, \cdots, t_e$, we take $(g_1, \cdots, g_k) = (t_1, \cdots, t_e)$ and $(f_1, \cdots, f_m) = (t_1, \cdots, t_e, s_1, \cdots, s_d)$. From the lemma we conclude immediately that

Proposition 4.2 *If $\pi : D \to S$ is an admissible deformation of D_0, and $\rho : \mathcal{D} \to S\times T$ is a free extension of π, then*

(1) provided $0 \leq k <$ codim $C_\pi^{\log}$, depth $\check{\Omega}_{D/S}^k \geq \mathrm{dim}D_0 - k$ and $d\pi : \check{\Omega}_{D/S}^k \to \check{\Omega}_D^{k+d}$ is an injection.

(2) If D_0 is an almost free divisor and $\pi : D \to S$ is a deformation which frees D_0, then codim $C_\pi^{\log} =$ dim $S + 1$, so that the assertions of (1) hold for $k \leq \mathrm{dim}D_0$.

(3) The map $\lambda d\pi : \check{\Omega}_{D/S}^k \to \check{\Omega}_{\mathcal{D}}^{k+e}/(t_1, \cdots, t_e)\check{\Omega}_{\mathcal{D}}^{k+e}$ defined by wedging with $ds_1 \wedge \cdots \wedge ds_d \wedge dt_1 \wedge \cdots \wedge dt_e$, is 1-1 for $0 \leq k <$ codim $C_\pi^{\log}$. □

Proposition 4.3 *Let $\pi : D \to S$ be an admissible deformation of the p-dimensional divisor D_0. Then*

1. *$\check{\Omega}_{D/S}^k$ is a torsion-free $\mathcal{O}_D$-module for $0 \leq k <$ codim $C_\pi^{\log}$ (in the sense that every non-zero-divisor in $\mathcal{O}_D$ is regular on $\check{\Omega}_{D/S}^k$);*
2. *$\check{\Omega}_{D/S}^k$ is a torsion module for $k =$ dim $D_0 + 1$; (and recall that $\check{\Omega}_{D/S}^k = 0$ for $k >$ dim $D_0 + 1$).*
3. *if $\rho : \mathcal{D} \to S\times T$ is a free extension of π then the cokernel of $\lambda d\pi : \check{\Omega}_{D/S}^p \to \check{\Omega}_{\mathcal{D}}^{p+d+e} \otimes \mathcal{O}_D$ is isomorphic to $T_{D/S}^{1,\log}$. This cokernel is independent of choice of ρ;*
4. *if D_0 is almost free then the torsion submodule of $\check{\Omega}_{D_0}^p$ has dimension equal to $\mathrm{dim}T_{D_0}^{1,\log}$.*

Proof 1. Let $p : \mathcal{D} \to S{\times}T$ be a free extension of π, with parameter space T of dimension e. Write $d = \dim S$. For $0 \leq k < \operatorname{codim} C_\pi^{\log}$, $\lambda d\pi : \check{\Omega}^k_{D/S} \to \check{\Omega}^{k+d}_D$ is an injection, so it remains only to show that $\check{\Omega}^{k+d}_D$ is torsion-free. This follows by the last part of Proposition 4.2: $dh\wedge$ embeds $\check{\Omega}^{k+d}_D$ in the free $\mathcal{O}_D$-module $\Omega^{k+d+1}_{V\times S}/h\Omega^{k+d+1}_{V\times S}$.

2. This is clear, since if $k > \dim D_0$ then $\check{\Omega}^k_{D/S}$ is supported on $C_\pi^{\log}$, whose dimension is less than that of D.

3. Contraction of the generator $dy\wedge ds\wedge dt$ of $\Omega^{p+d+e+1}_{V\times S\times T}$ by vector fields gives an isomorphism $\iota : \theta_{V\times S\times T} \to \Omega^{p+d+e}_{V\times S\times T}$ which restricts to an isomorphism $\mathrm{Der}(\log \mathcal{D}) \simeq h\Omega^{p+d+e}(\log \mathcal{D})$. Thus

$$\frac{\theta_{V\times S\times T}}{\mathrm{Der}(\log \mathcal{D})} \simeq \check{\Omega}^{p+d+e}_{\mathcal{D}}.$$

The preimage under ι of the submodule $dt\wedge ds\wedge \Omega^p_{V\times S\times T}$ of $\Omega^{p+d+e}_{V\times S\times T}$ is the module of vector fields with no component in the $S{\times}T$ direction, i.e. $\ker t\rho : \theta_{V\times S\times T} \to \theta(\rho)$; putting everything together, it follows that

$$\frac{\check{\Omega}^{p+d+e}_{\mathcal{D}}}{dt\wedge ds\wedge \check{\Omega}^p_{D/S}} \otimes \mathcal{O}_D \simeq \frac{\theta_{V\times S\times T}}{\mathrm{Der}(\log \mathcal{D}) + \ker(t\rho)} \otimes \mathcal{O}_D \simeq \frac{\theta(\rho)}{t\rho(\mathrm{Der}(\log \mathcal{D})} \otimes \mathcal{O}_D \simeq T^{1,\log}_{D/S}.$$

Independence of the choice of ρ follows as usual from Lemma 2.9.

4. Let $\pi : D \to S$ be a free deformation of D_0, and let $\dim S = d$. Then the torsion submodule of $\check{\Omega}^p_{D_0}$ is equal to the kernel of the map

$$ds_1\wedge\cdots\wedge ds_d : \check{\Omega}^p_{D_0} = \frac{\check{\Omega}^p_{D/S}}{(s_1,\cdots,s_d)\check{\Omega}^p_{D/S}} \to \frac{\check{\Omega}^{p+d}_D}{(s_1,\cdots,s_d)\check{\Omega}^{p+d}_D},$$

for composition with the injective morphism $dh\wedge$ maps $\check{\Omega}^p_{D_0}$ into the free $\mathcal{O}_{D_0}$-module $\Omega^{p+d+1}_{V\times S}/(h,s_1,\cdots,s_d)\Omega^{p+d+1}_{V\times S}$ (so that $T\check{\Omega}^p_{D_0}$ is contained in the kernel), while on the other hand the kernel is supported only at 0, since D_0 is free elsewehere, and hence is a torsion module. Now this kernel can be identified with $\mathrm{Tor}_1^{\mathcal{O}_S}(T^{1,\log}_{D/S}, \mathcal{O}_S/m_{S,0})$; for tensoring the short exact sequence

$$0 \to \check{\Omega}^p_{D/S} \xrightarrow{ds\wedge} \check{\Omega}^{p+d}_D \to T^{1,\log}_D/S \to 0$$

with $\mathbb{C} = \mathcal{O}_S/m_{S,0}$ we get the long exact sequence

$$0 = \mathrm{Tor}_1^{\mathcal{O}_S}(\check{\Omega}^{p+d}_D, \mathbb{C}) \to \mathrm{Tor}_1^{\mathcal{O}_S}(T^{1,\log}_{D/S}, \mathbb{C}) \to \check{\Omega}^p_{D_0} \to \check{\Omega}^{p+d}_D \otimes \mathbb{C} \to T^{1,\log}_{D_0} \to 0.$$

Now assume furthermore that $\pi : D \to S$ is versal (i.e. the inducing map $i : V\times S \to W$ is a $\mathcal{K}_{E,e}$-versal deformation of the map i_0 which induces D_0). Then there is an exact sequence

$$0 \to \mathcal{L} \xrightarrow{\alpha} \theta_S \xrightarrow{\rho} \pi_*(T^{1,\log}_{D/S}) \to 0 \qquad (2)$$

in which ρ is the Kodaira-Spencer map of the deformation. As $T^{1,\log}_{D/S}$ is a Cohen-Macaulay $\mathcal{O}_{V\times S}$-module of dimension $d-1$, by 2.10, and is finite over $\mathcal{O}_S$, it follows that its $\mathcal{O}_S$-depth is

equal to $d-1$ also, and it follows that $\mathcal{L}$ is free, of rank d. If π is miniversal, then all entries in the matrix α lie in $m_{S,0}$ and $\alpha \otimes \mathbb{C} = 0$. Hence $\mathrm{Tor}_1^{\mathcal{O}_S}(T^{1,\log}_{D/S}, \mathbb{C}) \simeq \mathcal{L}/m_{S,0}\mathcal{L}$ has dimension $d = \dim S = \dim T^{1,\log}_{D_0}$. $\square$

Part (4) of this result, together with Damon's theorem identifying the $\mathcal{A}_e$-normal space of a map-germ $f_0 : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ with $T^{1,\log}_{D(f_0)}$ (re-proved in Section 8 below), now proves Theorem 1.1.

Example 4.4 Suppose that $E \subset W$ is a weighted homogeneous free divisor and $i : V \to W$ is weighted homogeneous of $\mathcal{K}_{E,e}$-codimension 1. Let $D_0 = i^{-1}(E)$, let $x_1, \cdots, x_p$ be coordinates on V, and denote by χ_e the Euler vector field on V. Then the torsion in $\check{\Omega}^p_{D_0}$ is generated over $\mathbb{C}$ by the class in $\check{\Omega}^p_{D_0}$ of $\iota_{\chi_e}(dx_1 \wedge \cdots \wedge dx_p)$. For $\check{\Omega}^p_{D_0}$ is supported precisely at those points where i is not algebraically transverse to E, and thus $dx_1 \wedge \cdots \wedge dx_p$ is a torsion form in $\check{\Omega}^p_{D_0}$. As the Euler field is logarithmic, $\iota_{\chi_e}(dx_1 \wedge \cdots \wedge dx_p)$ is also a torsion form (the contraction is well-defined in $\check{\Omega}^{p-1}_{D_0}$ by 3.12). It is not zero, since its exterior derivative is a non-zero multiple of $dx_1 \wedge \cdots \wedge dx_p$.

5 Using $\check{\Omega}^\bullet_{D/S}$ to calculate cohomology

Definition 5.1 *The analytic space $D \subseteq W$ is* locally quasihomogeneous *if for each $x \in D$ there exist local analytic coordinates for W, centred at x, with respect to which the germ D, x is weighted homogeneous (with strictly positive weights).*

This property played an important rôle in [6] (under the name *strong quasihomogeneity*):

Theorem 5.2 [6] *Let $D \subset \mathbb{C}^n$ be a locally quasihomogeneous free divisor, and let $U = \mathbb{C}^n \setminus D$. Then integration along cycles induces an isomorphism*

$$h^q(\Gamma(\mathbb{C}^n, \Omega^\bullet(\log D))) \simeq H^q(U; \mathbb{C})$$

for $0 \leq q \leq n$. $\square$

Here and in what follows, we use the small h^q for the q'th homology of a complex.

One can speak of a locally quasihomogeneous germ - one which has a locally quasihomogeneous representative. This allows the apparent oxymoron of a quasihomogeneous germ which is not locally quasihomogeneous; however this incongruity seems less disturbing than the possibility of a strongly quasihomogeneous space which is not (globally) quasihomogeneous, so the term "locally quasihomogeneous" is to be preferred to "strongly quasihomogeneous".

By (3.3).6, if $D \subseteq V$ is a locally weighted homogeneous divisor then the complex

$$0 \to \mathbb{C}_D \to \mathcal{O}_D \to \check{\Omega}^1_D \to \check{\Omega}^2_D \to \cdots$$

is a resolution of $\mathbb{C}_D$. It follows that if V is a Stein space then

$$h^q(\Gamma(D, \check{\Omega}^\bullet_D)) \simeq H^q(D; \mathbb{C}).$$

Hence we can use the complex $\check{\Omega}^k_{D/S}$ to calculate the cohomology of the free fibres of an admissible deformation, provided the free fibres are locally quasihomogeneous.

Example 5.3 1. If (dim U, dim V) are in Mather's range of nice dimensions (cf [27]) then every stable map-germ $U \to V$ is quasihomogeneous with respect to some coordinate system. (In fact this characterises the nice dimensions). It follows that in the nice dimensions stable discriminants are locally quasihomogeneous. Thus, for our motivating example, the insistence on local quasihomogeneity is not unduly restrictive.

Definition 5.4 *Let $E \subseteq W$ be a divisor and let $\mathcal{S}$ be the canonical Whitney stratification of E.*

1. *A stratum in $\mathcal{S}$ is* locally quasihomogeneous *if at each point $x \in S$, there are local analytic coordinates centred at x with respect to which E is weighted homogeneous (with respect to strictly positive weights). The* weighted homogeneous codimension *$wh(E)$ of E is the infimum of the codimensions (in E) of strata of $\mathcal{S}$ which are not locally quasihomogeneous, and ∞ if E is locally quasihomogeneous.*
2. *A stratum $S \in \mathcal{S}$ is* holonomic *if at each point $x \in S$, $T_xS = T_x^{\log}E$. The* holonomic codimension *$hn(E)$of E is the infimum of the codimensions (in E) of strata of $\mathcal{S}$ which are not holonomic, and ∞ if every stratum is holonomic.*

Lemma 5.5 *Let D_0 be an almost free divisor based on E, suppose dim $D_0 \leq wh(E)$, and let $\pi : D \to S$ be an admissible deformation of D_0. Then for $x \notin C_\pi^{\log}$, the complex $\check{\Omega}^\bullet_{D/S}$ is a resolution of $\pi^{-1}(\mathcal{O}_S)$.*

Proof Let $i : V \times S \to W$ induce D from E (i.e. $D = i^{-1}(E)$.) and for $s \in S$ let $i_s : V \to W$ be defined by $i_s(x) = i(x, s)$. By a variant of the standard transversality lemma, if $(x, s) \notin C_\pi^{\log}$ the i_s is algebraically transverse to E at x, and thus transverse to each of the strata in the canonical Whitney stratification of E. As dim$V <$ wh(E), i_s meets only strata along which E is locally weighted homogeneous, and thus $D_s = i_s^{-1}(E)$ is locally weighted homogeneous. It follows that $\check{\Omega}^\bullet_{D_s,x}$ is a resolution of $\mathbb{C}_{D_s,x}$.

Again because $(x, s) \notin C_\pi^{\log}$, there is a commutative diagram

$$\begin{tikzcd} (D,(x,s)) \arrow[rr, "\simeq"] \arrow[dr, "\pi"'] & & (D_s,x)\times(S,s) \arrow[dl, "\text{projection}"] \\ & (S,s) & \end{tikzcd}$$

and the isomorphism $(D, (x, s)) \simeq (D_s, x)\times(S, s)$ induces an isomorphism of complexes

$$\check{\Omega}^\bullet_{D/S} \simeq \check{\Omega}^\bullet_{D_s} \otimes_{\mathbb{C}} \pi^{-1}(\mathcal{O}_S)$$

and thus $\check{\Omega}^\bullet_{D/S}$ is a resolution of $\mathbb{C} \otimes \pi^{-1}(\mathcal{O}_S) = \pi^{-1}(\mathcal{O}_S)$. □

Our notion of good representative of a germ of admissible deformation $\pi : (D, (x_0, 0)) \to S, 0$ of an almost free divisor D_0, x_0 is an obvious modification of the notion standard in the theory of isolated singularities, as described in some detail in [22], pp 25-26. We require an open set $X \subset \mathbb{C}^{p+1}\times\mathbb{C}^d$, a map $i : X \to W$ such that, writing $\tilde{D} = i^{-1}(E)$, we have $\tilde{D} \cap \mathbb{C}^{p+1}\times\{0\} = D$ (as germs at x_0), a real-analytic function $r : X \to [0, \infty)$ and a real $\varepsilon > 0$

such that $r^{-1}(0) \cap D_0 = \{x_0\}$ and such that $\tilde{D}_0$ is stratified transverse to $r^{-1}(\varepsilon')$ (in $\mathbb{C}^{p+1}$) for each ε' with $0 < \varepsilon' \leq \varepsilon$. Then there exists a contractible $S \subset \mathbb{C}^d$ containing 0 such that the restriction of π to each stratum of the canonical Whitney stratification of $\tilde{D}$ is a submersion at each point of $\pi^{-1}(S) \cap r^{-1}(\varepsilon)$. Finally, we take $A = r^{-1}([0, \varepsilon)) \cap \pi^{-1}(S)$, and $D = \tilde{D} \cap A$.

From now on we suppose that we are given a good Stein representative of an admissible deformation $\pi : D, (x_0, 0) \to S$ of the p-dimensional almost free divisor (D_0, x_0) based on the free divisor E. We suppose that $p < \min\{\mathrm{wh}(E), \mathrm{hn}(E)\}$, and denote the dimension of D by m. Consider the short exact sequence of complexes

$$0 \to \mathcal{H}^0(\check{\Omega}^\bullet_{D/S}) \to \check{\Omega}^\bullet_{D/S} \to \check{\Omega}^\bullet_{D/S,(>0)} \to 0 \tag{3}$$

where $\mathcal{H}^0(\check{\Omega}^\bullet_{D/S})$ is the 0'th cohomology sheaf (i.e. $\ker(d : \check{\Omega}^0_{D/S} \to \check{\Omega}^1_{D/S})$) and $\check{\Omega}^\bullet_{D/S,(>0)}$ is the complex $0 \to \check{\Omega}^0_{D/S}/\ker(d : \check{\Omega}^0_{D/S} \to \check{\Omega}^1_{D/S}) \to \check{\Omega}^1_{D/S} \to \cdots$. Applying the functor π_* to this sequence and deriving, we obtain a long exact sequence of cohomology on S:

$$\cdots \to R^q\pi_*(\mathcal{H}^0(\check{\Omega}^\bullet_{D/S})) \to \mathbb{R}^q\pi_*(\check{\Omega}^k_{D/S}) \to \mathbb{R}^q\pi_*(\check{\Omega}^\bullet_{D/S,(>0)}) \to \mathcal{R}^{q+1}\pi_*(\mathcal{H}^0(\check{\Omega}^\bullet_{D/S})) \to \cdots \tag{4}$$

for $q \geq 1$.

Lemma 5.6 *In these circumstances,*

(1) provided dim$V > 1$, $\mathcal{H}^0(\check{\Omega}^\bullet_{D/S}) = \pi^{-1}(\mathcal{O}_S)$, and so $R^q\pi_*(\mathcal{H}^0(\check{\Omega}^\bullet_{D/S}))$ is canonically isomorphic to $R^q\pi_*(\mathbb{C}_D) \otimes_{\mathbb{C}} \mathcal{O}_S$.

(2)

$$R^q\pi_*\check{\Omega}^\bullet_{D/S,(>0)} \simeq \pi_*(\mathcal{H}^q(\check{\Omega}^\bullet_{D/S,(>0)}) = \pi_*(\mathcal{H}^q(\check{\Omega}^\bullet_{D/S})$$

for $q \geq 1$.

(3)

$$R^q\pi_*(\check{\Omega}^\bullet_{D/S}) = \mathcal{H}^q(\pi_*\check{\Omega}^\bullet_{D/S}).$$

Proof (1) The statement is clearly true outside of $C_\pi^{\log}$, and moreover it is obvious that $\mathcal{H}^0(\check{\Omega}^\bullet_{D/S}) \supseteq \pi^{-1}(\mathcal{O}_S)$ everywhere. The opposite inclusion follows from the connectedness of the fibres of π.

(2) As D_0 is an almost free divisor, it follows that $\pi_{|C_\pi^{\log}}$ is finite. For $q > 0$, $\mathcal{H}^q(\check{\Omega}^\bullet_{D/S})$ is supported only on $C_\pi^{\log}$, and thus $\mathbb{R}^i\pi_*\mathcal{H}^q(\check{\Omega}^\bullet_{D/S}) = 0$ for $i > 0$. Thus the spectral sequence $\mathbb{R}^p\pi_*\mathcal{H}^q(\check{\Omega}^\bullet_{D/S,(>0)}) \Rightarrow R^{p+q}\pi_*\check{\Omega}^\bullet_{D/S,(>0)}$ collapses at E_1 and the conclusion follows.

3. This is an immediate consequence of the $\mathcal{O}_D$-coherence of the sheaves $\check{\Omega}^k_{D/S}$, together with the fact that we have chosen a Stein representative. □

Corollary 5.7 *Provided dim $V > 1$, the long exact sequence (4) reduces to*

$$\cdots \to R^q\pi_*(\mathbb{C}_D) \otimes \mathcal{O}_S \to \mathcal{H}^q(\pi_*(\check{\Omega}^k_{D/S})) \to \pi_*(\mathcal{H}^q(\check{\Omega}^\bullet_{D/S})) \to \mathcal{R}^{q+1}\pi_*\mathbb{C}_D \otimes \mathcal{O}_S \to \cdots \tag{5}$$

for $q > 1$. □

Theorem 5.8 *Let $\pi : D \to S$ be a good representative of an admissible deformation of the p-dimensional almost free divisor D_0, and suppose that $p < hn(E)$. Then $\mathcal{H}^q(\pi_*(\check{\Omega}^k_{D/S}))$ is a coherent sheaf of $\mathcal{O}_S$-modules.*

Proof We use a theorem of Duco van Straten ([32]). Let $\pi : D \to S$ be a good representative of an admissible deformation of an almost free divisor, and let $\mathcal{L}$ be a sheaf of complex vector spaces on D. $\mathcal{L}$ is *transversally constant* at the boundary if there exist an open neighbourhood U of ∂A and a C^∞ vector field θ on U with the following properties:

1. θ is transverse to ∂A;
2. θ is tangent to D and to the fibres of π;
3. the restriction of $\mathcal{L}$ to each local integral curve of θ is a constant sheaf.

Then van Straten proves

Theorem 5.9 *Let $\pi : D \to S$ be a good representative of a germ $\pi : (D, 0) \to (S, 0)$, and let $K^\bullet$ be a finite complex of sheaves on D. Assume*

1. *the sheaves K^i are $\mathcal{O}_D$-coherent;*
2. *the differentials $K^i \to K^{i+1}$ are $\pi^{-1}(\mathcal{O}_S)$-linear;*
3. *the cohomology sheaves $\mathcal{H}^i(K^\bullet)$ are transversally constant at the boundary.*

Then $\mathbb{R}^i\pi_* K^\bullet$ is an $\mathcal{O}_S$-coherent module. □

The statement in [32] assumes a 1-dimensional base S, but this hypothesis is not used in the proof.

Lemma 5.10 *Let $\pi : D \to S$ be a good representative of an admissible deformation of the p-dimensional almost free divisor D_0 based on the free divisor E, with $p < hn(E)$. Then the cohomology sheaves $\mathcal{H}^i(\check{\Omega}^\bullet_{D/S})$ are transversally constant at the boundary.*

Proof As $C^{\log}_\pi \cap D_0 = \{(0, 0)\}$, at each point of the boundary $\partial A \cap D_0$ all the strata of the canonical Whitney stratification $\mathcal{X}_0$ of D_0 are holonomic, i.e. for each stratum $X \in \mathcal{X}_0$ and $x \in \partial D$, $T_x X = T^{\log}_x D_0$. Thus, since ∂A_0 is transverse to $\mathcal{X}_0$, at each point $x \in \partial D_0$ there exists a germ of vector field $\chi^0_x \in \mathrm{Der}(\log D_0)_x$ such that $\chi^0_x(x) \notin T_x \partial A$. As π is locally trivial at $(x, 0)$, the logarithmic germ χ^0_x extends to a germ $\chi_x \in \mathrm{Der}(\log D)_{(x,0)} \cap \ker(d\pi)$; there is a neighbourhood U_x of $(x, 0)$ in $V \times S$ such that for all $(x', s) \in \partial D \cap U_x$, $\chi_x(x', s) \notin T_{(x,s)} \partial A$. Let $U = \bigcup_{x \in D_0 \cap \partial A} U_x$. Now by means of a partition of unity subordinate to the open cover $\{U_x\}$ of U, we construct from these χ_x a C^∞ vector field θ on U such that

1. for all $(x, s) \in U$, $\theta(x, s) \in T^{\log}_{(x,s)} D$
2. for all $(x, s) \in U \cap \partial D$, $\theta(x, s) \notin T_{(x,s)} \partial A$.

By shrinking U if necessary, we can suppose that $U \cap C_\pi^{\log} = \emptyset$. Thus at each point $(x,s) \in U$, π is locally analytically trivial. As θ is tangent to the fibres of π and logarithmic, it follows that the sheaves $\mathcal{H}^i(\check{\Omega}^\bullet_{D/S})$ are constant along the integral curves of θ. $\square$

Coherence of the sheaves $\mathbb{R}^q\pi_*\check{\Omega}^\bullet_{D/S}$ now follows, by van Straten's theorem. $\square$

As in the classical case, that is, the theory of isolated complete intersection singularities (ICIS), from the coherence theorem follow a number of interesting results. In most cases proofs are identical with those of the corresponding classical results in [22], and are omitted where possible.

Following Brieskorn and Greuel, we define three associated modules. Let $\pi : D \to S$ be a good representative of an admissible deformation, and let $\rho : \mathcal{D} \to S \times T$ be a free extension. We write $\dim S = d,\ \dim T = r$. For brevity let us denote by $\check{\mathcal{H}}$ the module $\mathcal{H}^p(\pi_*\check{\Omega}^\bullet_{D/S})$.

$$\check{\mathcal{H}}' = \frac{\pi_*\check{\Omega}^p_{D/S}}{d(\pi_*\check{\Omega}^{p-1}_{D/S})};$$

$$\check{\mathcal{H}}'' = \frac{\pi_*\check{\Omega}^m_D}{ds \wedge d(\pi_*\check{\Omega}^{p-1}_{D/S})}.$$

$$\check{\mathcal{H}}''' = \frac{\pi_*(\check{\Omega}^{p+d+r}_{\mathcal{D}})}{\lambda ds \wedge d(\pi_*\check{\Omega}^{p-1}_{D/S})}.$$

Note that by Lemma 2.9, $\check{\mathcal{H}}'''$ does not depend on the choice of free extension ρ of π. We also define

$$\check{H} = \mathcal{H}^p(\check{\Omega}^\bullet_{D/S,(x_0,0)});$$

$$\check{H}' = \frac{\check{\Omega}^p_{D/S,(x_0,0)}}{d\check{\Omega}^{p-1}_{D/S,(x_0,0)}};$$

$$\check{H}'' = \frac{\check{\Omega}^m_{D,(x_0,0)}}{d\pi \wedge d\check{\Omega}^{p-1}_{D/S,(x_0,0)}}.$$

$$\check{H}''' = \frac{\check{\Omega}^{p+d+r}_{\mathcal{D},(x_0,0)}}{\lambda ds \wedge d\check{\Omega}^{p-1}_{D/S,(x_0,0)}}.$$

By a standard argument (see e.g [22] 8.6), if $\pi : D \to S$ is a good representative of $\pi : (D,(x_0,0)) \to S, 0$ then the stalks at $0 \in S$ of $\check{\mathcal{H}}, \check{\mathcal{H}}', \check{\mathcal{H}}''$ and $\check{\mathcal{H}}'''$ are canonically isomorphic to $\check{H}, \check{H}', \check{H}''$ and $\check{H}'''$ respectively.

Proposition 5.11 *Let D_0, x_0 be a p-dimensional almost free divisor based on the free divisor E with $p < hn(E)$, and let $\pi : D \to S$ be a good Stein representative of an admissible deformation of D_0, x_0. Then $\check{\mathcal{H}}', \check{\mathcal{H}}''$ and $\check{\mathcal{H}}'''$ are $\mathcal{O}_S$-coherent.*

Proof Apply the argument of 5.8 to the complexes $\check{\Omega}^\bullet_{D/S,(\leq p)}$, $d\pi(\check{\Omega}^\bullet_{D/S,(\leq p)})$ and $\lambda d\pi(\check{\Omega}^\bullet_{D/S,(\leq p)})$. $\square$

Proposition 5.12 *Let D, x_0 be a p-dimensional almost free divisor based on the free divisor E, with wh(E) $> p+1$. Then the complex*

$$0 \to \mathbb{C}_{D,x_0} \to \mathcal{O}_{D,x_0} \to \check{\Omega}^1_{D,x_0} \to \cdots \to \check{\Omega}^p_{D,x_0}$$

is exact, $H^0_{\{x_0\}}(d\check{\Omega}^{p-1}_{D,x_0}) = 0$, and there is a natural chain of injections

$$H^1_{\{x_0\}}(d\check{\Omega}^{p-2}_D) \hookrightarrow H^2_{\{x_0\}}(d\check{\Omega}^{p-3}_D) \hookrightarrow \cdots \hookrightarrow H^{p-1}_{\{x_0\}}(d\mathcal{O}_D) \hookrightarrow H^p_{\{x_0\}}(\mathbb{C}_D).$$

Proof Looijenga's proof of the corresponding result ([22]8.19) in the case of a deformation $f : X \to S$ of an ICIS applies without change in this context. It relies only on estimates of the depth of the modules $\mathcal{O}^q_{X,x_0}$, and on the acyclicity of $\Omega^\bullet_{X,x_0}$ off x_0. Our depth estimates here (4.2) are formally identical to the estimates in the classical case (compare 8.15 and 8.16 of [22]), and the hypothesis that $p + 1 < \mathrm{wh}(E)$ guarantees the acyclicity of $\check{\Omega}^\bullet_D$ off $C^{\log}_\pi$. $\square$

From this result follows a similar statement in the relative case:

Proposition 5.13 *Suppose that $\pi : D \to S$ is an admissible deformation of the p-dimensional almost free divisor D_0, x_0 based on the free divisor E, and that $p + 1 < \min\{wh(E), hn(E)\}$. Then $\check{H}'$, and $\check{H}'''$ are free $\mathcal{O}_{S,0}$ modules of rank $\mu_\Delta(D_0, x_0)$, and if π is a deformation which frees D_0, then the complex*

$$0 \to \mathcal{O}_{S,0} \xrightarrow{f^*} \mathcal{O}_{D/S,(x_0,0)} \to \check{\Omega}^1_{D/S,(x_0,0)} \to \cdots \to \check{\Omega}^p_{D/S,(x_0,0)}$$

is exact.

Proof Again, Looijenga's proof of the corresponding result, [22] 8.20, applies practically verbatim. The proof is by induction on dim S, but does not involve fibrations with fibre dimension greater than p. There is a minor difference of notation: Looijenga uses ω_f in place of Greuel's H''', to which our notation $\check{H}'''$ refers implicitly. $\square$

6 The Gauss-Manin connection

Let $\pi : D \to S$ be a good Stein representative of a deformation which frees the p-dimensional almost free divisor D_0, x_0, and let $\mathcal{B} \subset S$ be the logarithmic discriminant.

The sheaf $\mathbb{R}^p\pi_*\mathbb{C}_D$ is a local system off $\mathcal{B}$, and so $\mathbb{R}^p\pi_*\mathbb{C}_D \otimes \mathcal{O}_S$ is naturally endowed with a flat connection. Because of the natural isomorphism

$$\mathbb{R}^p\pi_*\mathbb{C}_D \otimes \mathcal{O}_S \simeq \mathcal{H}^p(\pi_*\check{\Omega}^p_{D/S})$$

off $\mathcal{B}$, $\mathcal{H}^p(\pi_*\check{\Omega}^p_{D/S})$ thus has a natural flat connection off $\mathcal{B}$, the topological connection. In [22] the extension of the corresponding flat connection on the vanishing cohomology of the Milnor

fibration $f : X \to S$ of an ICIS to the singular Gauss-Manin connection on $H^p(\Omega^\bullet_{X/S,(x_0,0)})$ is described in terms of Lie derivatives; the same construction serves in the case of an admissible deformation of an almost free divisor. Following Looijenga's account, we now sketch this.

First, if $s \notin \mathcal{B}$, then every vector field germ $\eta \in \theta_{S,s}$ has a lift to a section ξ of $\pi_*\mathrm{Der}(\log D)_s$; as in [22], this is a consequence of the fact that $\pi : D \to S$ is a Stein map. For $[\omega] \in \mathcal{H}^p(\pi_*\check{\Omega}^\bullet_{D/S})_s$, set

$$\nabla_\eta([\omega]) = [L_\xi(\omega)],$$

where L_ξ is the Lie derivative. Note that Cartan's formula $L_\xi(\omega) = d\iota_\xi(\omega) + \iota_\xi(d\omega)$ holds for $\omega \in \check{\Omega}^p_D$, since both the exterior derivative d and contraction by the logarithmic vector field ξ preserve the subcomplex $h\Omega^\bullet(\log D)$ of $\Omega^\bullet_A$ (where A is the ambient Milnor ball). It is easy to check that L_ξ is well defined on the relative complex $\Omega^\bullet_{D/S}$ and commutes with its exterior derivative.

If ξ' is another lift of η then $\xi - \xi'$ is vertical with respect to π, so since $d\omega \in \pi_*(\pi^*\Omega^1_{S,s} \wedge \check{\Omega}^p_D)_s$, we have

$$\begin{aligned} L_{\xi-\xi'}(\omega) &= \iota_{\xi-\xi'}(d\omega) + d(\iota_{\xi-\xi'}(\omega)) \\ &= d(\iota_{\xi-\xi'}(\omega)). \end{aligned}$$

Thus $\nabla : \mathcal{H}^p(\pi_*\check{\Omega}^\bullet_{D/S})_s \to \mathcal{H}^p(\pi_*\check{\Omega}^\bullet_{D/S})_s \otimes_{\mathcal{O}_S} \Omega^1_{S,s}$ is well defined.

Looijenga's proof that ∇ so defined coincides with the topological connection works verbatim: one needs only to check that the pairing $\mathcal{H}^p(\pi_*\check{\Omega}^\bullet_{D/S,s}) \times H_p(D_s;\mathbb{C}) \to \mathcal{O}_{S,s}$ (given by integrating forms over the translation of cycles $Z \in D_s$ to nearby fibres by means of the local trivialisation of the fibration $\pi : D \to S$ around s) is non-degenerate in $\mathcal{H}^p(\pi_*\check{\Omega}^\bullet_{D/S,s})$, and the argument on pages 149-150 of [22] shows this, without any modification.

The connection ∇ extends to a meromorphic connection on $\mathcal{H}^p(\pi_*\check{\Omega}^\bullet_{D/S,s})$ for points $s \in \mathcal{B}$, as follows: let g be a defining equation for the hypersurface $\mathcal{B}$, and let $\mathcal{L}$ denote the (coherent) $\mathcal{O}_S$ module of π-liftable vector fields. If $\eta \in \theta_{S,s}$ then for some positive power m, $g^m\eta$ is liftable, by virtue of the coherence of $\theta_S/\mathcal{L}$. Let ξ be a lift. Moreover, by the coherence of $\check{\mathcal{H}}'$ and $\mathcal{H}^p(\pi_*\check{\Omega}^\bullet_{D/S})$, and the fact that they coincide off $\mathcal{B}$, some power of g pushes $L_\xi(\omega)$ into $Z^p(\pi_*\check{\Omega}^\bullet_{D/S,s})$. Thus we can define

$$\nabla : \mathcal{H}^p(\pi_*\check{\Omega}^\bullet_{D/S}) \to \mathcal{H}^p(\pi_*\check{\Omega}^\bullet_{D/S})[g^{-1}] \otimes \Omega^1_S$$

by

$$\nabla_\eta([\omega]) = g^{-m}[L_\xi(\omega)]$$

where ξ is a lift of $g^m\eta$ and m is a sufficiently high power for the expression on the right to make sense.

Finally, this connection extends by Leibniz's rule to a meromorphic connection on $\mathcal{H}^p(\pi_*\check{\Omega}^\bullet_{D/S,s})$.

Theorem 6.1 *The Gauss-Manin connection just defined has a regular singularity along $\mathcal{B}$.*

Proof Once again, the proof for the classical case given in [22] applies verbatim. $\square$

7 Calculating μ_E

Lemma 7.1 *If D_0, x_0 is a p-dimensional almost free divisor based on the free divisor E, and $p < \min\{wh(E), hn(E)\}$, then $\mu_E(D_0, x_0) = dim\ \check{\Omega}^p_{D_0,x_0}/d\check{\Omega}^{p-1}_{D_0,x_0}$.*

Proof Let $\pi : D \to S$ be a deformation which frees D_0, x_0 (such deformations exist, by the hypothesis that $p < \mathrm{hn}(E)$). Then $\check{H}' = \check{\Omega}^p_{D/S,(x_0,0)}/d\check{\Omega}^{p-1}_{D/S,(x_0,0)}$ is a free $\mathcal{O}_{S,0}$-module of rank μ_E, and $\check{H}'/m_{S,0}\check{H}' = \check{\Omega}^p_{D_0,x_0}/d\check{\Omega}^{p-1}_{D_0,x_0}$. □

Proposition 7.2 *Suppose that D_0 is a p-dimensional almost free divisor based on the free divisor E, and that $\pi : D \to S$ is a deformation over the 1-dimensional base S, which frees (D_0, x_0), with the property that $(D, (x_0, 0))$ is also almost free. Then provided $p + 1 < wh(E)$, $\mu_E(D_0) + \mu_E(D) = dim\ \check{\Omega}^{p+1}_{D/S}$*

Proof The proof is practically identical to Greuel's proof of the corresponding result (Lemma 5.3) of [19]. Consider the map

$$\nabla'_{d/ds} : ds \wedge \check{H}' \to \check{H}'', \quad [ds \wedge \omega] \mapsto [d\omega].$$

Both $ds \wedge \check{H}'$ and $\check{H}''$ are $\mathbb{C}\{s\}$-modules of rank $\mu_E(D_0)$, with $ds \wedge \check{H}' \subseteq \check{H}''$, and

$$\nabla'_{d/ds}(gds \wedge [\omega]) = g.\nabla'_{d/ds}(ds \wedge [\omega]) + (dg/ds)\nabla'_{d/ds}(ds \wedge [\omega])$$

for $g \in \mathbb{C}^s$; it follows by the Malgrange index theorem (see for example [23],Theorem 2.3) that

$$\dim\ker\nabla'_{d/ds} - \dim\ \mathrm{coker}\nabla'_{d/ds} = \mathrm{rank}\ ds \wedge \check{H}' - \dim\ (\check{H}''/ds \wedge \check{H}').$$

Now $\nabla'_{d/ds}$ is injective (by 5.12) and so this equality amounts to

$$-\dim\ \frac{\check{\Omega}^{p+1}_{D,x}}{d\check{\Omega}^p_{D,x}} = \mu_E(D_0) + \dim\ \check{\Omega}^{p+1}_{D/S},$$

and thus to the required equality. □

Lemma 7.3 *$dim\ \check{\Omega}^{p+1}_{D/S,x} = dim T^{1,\log}_{D/S,x}$.*

Proof Let $\rho : \mathcal{D} \to S \times T$ be a free extension of $\pi : D \to S$. By the proof of 4.3(4), $T\check{\Omega}^{p+1}_{D,x} = \ker\ (\lambda : \check{\Omega}^{p+1}_{D,x} \to \check{\Omega}^{p+d+1}_{\mathcal{D},x})/(t)\check{\Omega}^{p+d+1}_{\mathcal{D},x}$ As $\lambda d\pi : \check{\Omega}^p_{D/S,x} \to \check{\Omega}^{p+d+1}_{\mathcal{D},x}$ is injective, $T\check{\Omega}^{p+1}_{D,x} \cap d\pi(\check{\Omega}^p_{D/S,x}) = 0$, and hence there is an exact sequence

$$0 \to T\check{\Omega}^{p+1}_{D,x} \to \check{\Omega}^{p+1}_{D/S,x} \to \frac{dt \wedge \check{\Omega}^{p+1}_{D,x}}{dt \wedge d\pi \wedge \check{\Omega}^p_{D/S}} \to 0.$$

From the exact sequence

$$0 \to \frac{dt \wedge \check{\Omega}^{p+1}_{D,x}}{dt \wedge d\pi \wedge \check{\Omega}^{p}_{D/S}} \to T^{1,\log}_{D/S} \to T^{1,\log}_{D} \to 0$$

and the fact that dim $T\check{\Omega}^{p+1}_{D,x} = \dim T^{1,\log}_{D}$ (by 4.3) we conclude that $\dim \check{\Omega}^{p+1}_{D/S,x} = \dim T^{1,\log}_{D/S,x}$. $\square$

Corollary 7.4 *With the hypotheses of 7.2, $\mu_E(D_0) + \mu_E(D) = dim_{\mathbb{C}}\ T^{1,\log}_{D/S}$.* $\square$

Exactly as in [19], this furnishes us with a means of calculating $\mu_E(D_0)$: we suppose that we are given a free deformation $\pi : D \to S$ of D_0 with the property that if $D_i = D \cap \{s_{i+1} = \cdots = s_d = 0\}$ then each germ $(D_i, 0)$ is almost free, and with $\dim D < \mathrm{wh}(E)$. We apply 7.4 to the deformations $\pi_i : D_i \to S_i$, where S_i is 1-dimensional and $\pi_i(x, s) = s_i$.

Corollary 7.5

$$\mu_E(D_0, 0) = \sum_{i=1}^{d} (-1)^{i+1} dim_{\mathbb{C}}(\check{\Omega}^{p+i}_{D_i/S_i,0})$$

$$= \sum_{i=1}^{d} (-1)^{i+1} dim_{\mathbb{C}}(T^{1,\log}_{D_i/S_i}).$$

$\square$

This should be contrasted with the calculation of μ_E given in [12], which we now summarise. Let $\pi : (D, (x_0, 0)) \to (S, 0)$ be a free deformation of the almost free divisor $D_0, x_0 = i^{-1}(E)$, suppose that π frees D_0, and suppose that we are given a defining equation h for D in the smooth ambient space A, with the property that there exists a vector field $\chi \in \theta_A$ such that $\chi.h = h$ (such an equation is called a "good defining equation" in [12]). Let $\mathrm{Der}(\log h)$ be the set of vector fields annihilating h - i.e., tangent to all of the level sets of h. Then we have

Proposition 7.6 *In these circumstances*

$$\mu_E(D_0, x_0) = dim_{\mathbb{C}}\ \frac{\theta(i)}{ti(\theta_V) + i^*(Der(\log h))}$$

$$= dim_{\mathbb{C}}\ \frac{\theta(\pi)}{t\pi(Der(\log h)) + m_{S,0}\theta(\pi)}.$$

$\square$

In fact [12] proves only the first of these equalities. The second follows, by the argument of the proof of 2.7.

Our 7.4 and 7.5 do not reduce in any obvious way to 7.6. For example, consider the case of a free and freeing deformation $\pi : D \to S$ with $\dim S = 1$. Then 7.4 gives

$$\mu_E(D_0, x_0) = \dim \frac{\theta(\pi)}{t\pi(\mathrm{Der}(\log D))} \tag{6}$$

while 7.6 gives

$$\mu_E(D_0, x_0) = \dim \frac{\theta(\pi)}{t\pi(\mathrm{Der}(\log h)) + m_{S,0}\theta(\pi)}. \tag{7}$$

Moreover, 7.4 does not require a good defining equation.

In fact the modules on the right hand side of (6) and (7) coincide when D is weighted homogeneous and the variable s has non-zero weight w_s: if $\chi_e = w_s s\partial/\partial s + \sum_i w_i x_i \partial/\partial x_i$ is the Euler vector field, then $\mathrm{Der}(\log D) = \mathrm{Der}(\log h) \oplus \langle \chi_e \rangle$, and $t\pi(\mathrm{Der}(\log D)) = t\pi(\mathrm{Der}(\log h)) + m_{S,0}\theta(\pi)$.

Another application of 7.4 is the following:

Corollary 7.7 *If D_0 is a p-dimensional almost free divisor based on the free divisor E, if $p < \min\{wh(E), hn(E)\}$, and if $\mu_E(D_0) = 1$, then the $\mathcal{K}_E$-discriminant in a versal deformation of D_0 is reduced.*

Proof Let i_0 induce D_0 from E. By the main theorem of [12], the $\mathcal{K}_{E,e}$-codimension of i_0 is no greater than $\mu_E(D_0)$; it is therefore equal to 1. Let i be a miniversal deformation of i_0 over the smooth 1-dimensional base S, and let $D = i^{-1}(E)$.

The $\mathcal{K}_E$ discriminant consists simply of $\{0\}$, but with analytic structure provided by the 0-th Fitting ideal of $T^{1,\log}_{D/S}$ as $\mathcal{O}_S$-module. By 7.4, $\mu_E(D_0) = \dim_{\mathbb{C}} T^{1,\log}_{D/S}$ (since i is algebraically transverse to E). Therefore $T^{1,\log}_{D/S}$ has length 1. It follows that its zero'th Fitting ideal over $\mathcal{O}_S$ is the maximal ideal $m_{S,0}$; hence the $\mathcal{K}_E$-discriminant is reduced. $\square$

We remark that the same argument shows that the vector field $s\partial/\partial s$ on S is liftable to a vector field $\chi \in \mathrm{Der}(\log D)$. See [10] for some striking results on the question of the reducedness and freeness of the $\mathcal{K}_E$-discriminant.

8 Discriminants of Maps and Damon's theorem

In this section we describe the relation between singularities and deformations of map-germs $f : \mathbb{C}^n, 0 \to \mathbb{C}^p, 0$ and the theory of almost free divisors.

In [8], Damon proves the following theorem:

Theorem 8.1 *Suppose that the diagram*

$$\begin{array}{ccc} \mathcal{U}, 0 & \xrightarrow{F} & \mathcal{V}, 0 \\ j_0 \uparrow & & i_0 \uparrow \\ U, 0 & \xrightarrow{f} & V, 0 \end{array}$$

is a fibre square (with i_0 transverse to F), and that F is stable. Let D be the discriminant of F (or image if $n < p$). Then

$$T^1_f \simeq \frac{\theta(i_0)}{ti_0(\theta_{\mathbb{C}^p}) + i_0^*(Der(\log D))}.$$

Here T^1_f is the $\mathcal{O}_{\mathbb{C}^p,0}$-module

$$\frac{\theta(f)}{tf(\theta_{\mathbb{C}^n}) + \omega f(\theta_{\mathbb{C}^p})},$$

also known as $N\mathcal{A}_e f$, and all the spaces are assumed non-singular.

Damon's proof in [8] used the additional hypothesis that f has finite $\mathcal{A}_e$-codimension, and was rather involved. C.T.C. Wall gave a simpler proof in [34]; here we give a proof along similar lines. Other proofs can be found in [17] and [29].

Proof of 8.1 Let us first assume that F is an unfolding of f, with $\mathcal{U} = U \times S, \mathcal{V} = V \times S$, and that $i_0 : V, 0 \to \mathcal{V}, 0$ is the standard inclusion $v \mapsto (v, 0)$.
Step 1

$$\frac{\theta(F)}{tF(\theta_{U\times S})} \simeq \frac{\theta(F/\pi)}{tF(\theta_{U\times S/S})}.$$

Step 2 By a standard argument (see e.g. [22] 6.14), $\mathrm{Der}(\log D)$ is the kernel of the morphism $\omega F : \theta_{V\times S} \to F_*(\theta(F)/tF(\theta_{U\times S}))$. There is a commutative diagram

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & & & & & \\
 & & \uparrow & & \uparrow & & & & & & \\
0 & \to & t\pi(\mathrm{Der}(\log D)) & \to & \theta(\pi) & \to & \frac{\theta(\pi)}{t\pi(\mathrm{Der}(\log D))} & \to & 0 & & \\
 & & t\pi \uparrow & & t\pi \uparrow & & & & & & \\
0 & \to & \mathrm{Der}(\log D) & \to & \theta_{V\times S} & \to & F_*\left(\frac{\theta(F)}{tF(\theta_{U\times S})}\right) & \to & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow\simeq & & & & \\
0 & \to & \mathrm{Der}(\log D)_\pi & \to & \theta_{V\times S/S} & \to & F_*\left(\frac{\theta_\pi(F)}{tF(\theta_{U\times S/S})}\right) & \to & T^1_{F/S} & \to & 0 \\
 & & \uparrow & & \uparrow & & & & & & \\
 & & 0 & & 0 & & & & & &
\end{array}$$

Diagram chasing gives an isomorphism

$$\frac{\theta(\pi)}{t\pi(\mathrm{Der}(\log D))} \simeq T^1_{F/S};$$

so now tensoring with $\mathcal{O}_{V,0}$ (over $\mathcal{O}_{V\times S,0}$) produces an isomorphism

$$\frac{\theta(\pi)}{t\pi(\mathrm{Der}(\log D)) + m_S\theta(\pi)} = \frac{\theta(\pi)}{t\pi(\mathrm{Der}(\log D))} \otimes \mathcal{O}_{V,0} \simeq T^1_f.$$

But

$$\frac{\theta(\pi)}{\pi^* m_{S,0}\theta(\pi)}$$

can be identified with

$$\frac{\theta(i_0)}{ti_0(\theta_V)}$$

and using this identification we obtain

$$\frac{\theta(i_0)}{ti_0(\theta_V) + i_0^*\mathrm{Der}(\log D)} \simeq \frac{\theta(\pi)}{t\pi(\mathrm{Der}(\log D)) + m_{S,0}\theta(\pi)} \simeq T^1_f.$$

This completes the proof in the special case where F is a (parametrised) unfolding of f. The general case follows by showing first that up to isomorphism, the module

$$\frac{\theta(i)}{ti(\theta_V) + i_0^*\mathrm{Der}(\log D)}$$

is independent of the choice of stable map F from which f is pulled back by i. For stable maps are classified by their local algebras ([25]), and the local algebra of F is isomorphic to that of f; hence, if $F_1 : \mathcal{U}_1, 0 \to \mathcal{V}_1, , 0$ and and $F_2 : \mathcal{U}_2, 0 \to \mathcal{V}_2, 0$ are stable maps from which f may be induced by transverse pull-back, with $\dim \mathcal{U}_1 \geq \dim \mathcal{U}_2$, then F_1 is equivalent to a trivial unfolding of F_2 on $\dim \mathcal{U}_1 - \dim \mathcal{U}_2$ parameters. As a consequence of this, we may assume that F is a stable unfolding of f, as in the special case. $\square$

Note that the isomorphism $T^1_{F/S} \simeq \theta(\pi)/t\pi(\mathrm{Der}(\log D))$ identifies the support of $T^1_{F/S}$ as the logarithmic critical space of $\pi : D \to S$. Note also that there is now a strong formal analogy between T^1_f and $T^1_{X_0}$, where $X_0 = f^{-1}(0)$:

$$T^1 f = \frac{\theta(\pi)}{t\pi(\mathrm{Der}(\log D)) + m_{\mathbb{C}^d,0}\theta(\pi)}$$

$$T^1 X_0 = \frac{\theta(f)}{tf(\theta_{\mathbb{C}^n,0}) + m_{\mathbb{C}^p,0}\theta(f)}.$$

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